A new numerical scheme for improved Businnesque equations with surface pressure

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In this work an improved Boussinesq model with a surface pressure term is discretized by a new approach. By specifying a single parameter the proposed discretization enables the user to run the program either in the long wave mode without dispersion terms or in the Boussinesq mode. Furthermore, the Boussinesq mode may be run either in the classical Boussinesq mode or in the improved Boussinesq mode by setting the dispersion parameter appropriately. In any one of these modes it is possible to specify a fixed or a moving surface pressure for simulating a moving object on the surface. The numerical model developed here is first tested by comparing the numerically simulated solitary waves with their analytical counterparts. The second test case concerns the comparison of the numerical solutions of moving surface pressures with the analytical solutions of the long wave equations for all possible modes (long wave, classical, and improved Boussinesq).

1 INTRODUCTION

The earliest depth-averaged wave model that included weakly dispersive and nonlinear effects was derived by Boussinesq (1871), in which the non-hydrostatic pressure was linearized and included in the momentum equations. The original equations were derived for constant depth only. Later, Mei and LeMéhauté [1], Peregrine [2] derived Boussinesq equations for variable depth. While Mei and LeMéhauté used the velocity at the bottom as the dependent variable, Peregrine used the depth-averaged velocity and assumed the vertical velocity varying linearly over the depth. Due to wide popularity of the equations derived by Peregrine, these equations are often referred to as the standard Boussinesq equations for variable depth in the coastal engineering community. The standard Boussinesq equations are valid only for relatively small \( kh \) and \( H/h \) values where \( kh \) and \( H/h \) represents the parameters indicating the relative depth (dispersion) and the wave steepness (nonlinearity), respectively. Madsen et al [3] and Madsen and Sørensen [4] included higher order terms with adjustable coefficients into the standard Boussinesq equations for constant and variable water depth, respectively. Beji and Nadaoka [5] gave an alternative derivation of Madsen et al’s [4] improved Boussinesq equations. Liu & Wu [7] presented a model with specific applications to ship waves generated by a moving pressure distribution in a rectangular and trapezoidal channel by using boundary integral method. Torsvik [9] presented a numerical investigation on waves generated by a pressure disturbance moving at constant speed in a channel with a variable cross-channel depth profile by using Lynett et al [8] and Liu & Wu [7]'s COULWAVE long wave model. The surface disturbance may come from a moving free surface object, bottom movement, or a moving object in between. The first case is associated with a moving surface pressure which is the main problem to be investigated in this study using Beji and Nadaoka's [5] alternative derivation. First of all the numerical model is verified for different test cases, such as comparing the numerically simulated solitary waves with the analytical solutions of the long wave equations for all possible modes (long wave, classical, and improved Boussinesq).

2 IMPROVED BOUSSINESQ EQUATIONS

Dispersion relation of Peregrine’s system [2] is an accurate approximation to Stokes first order wave theory for very small values of the dispersion parameter \( \mu \). Madsen et al [3] improved dispersion characteristics of this system by adding extra dispersive terms to the momentum equations as expressed in terms of depth integrated velocities \( P = (h + \eta)\overline{u} \) and \( Q = (h + \eta)\overline{v} \). The form of the dispersion relation is determined by matching the
dispersion characteristics to linear wave theory. Later, this procedure has been extended to the case of variable depth by Madsen and Sørensen [4]. Alternatively, Beji and Nadaoka [5] introduced a slightly different method to improve the dispersion characteristics by a simple algebraic manipulation of Peregrine’s work for variable depth.

2.1 Derivation of Beji and Nadaoka’s improved Boussinesq equations

Following the procedure given by Peregrine [2] the continuity and momentum equations are,

$$\frac{\partial \bf{u}}{\partial t} + (\bf{u} \cdot \nabla) \bf{u} + g \nabla \eta = \left(1 + \beta \frac{1}{2} \right) \nabla \cdot (h \bf{u}) \right]$$

$$+ \beta \frac{1}{6} \nabla \cdot (h \nabla \eta) - \left(1 + \beta \right) \frac{1}{6} \nabla \cdot (h \nabla \bf{u})$$

where $\beta$ is a scalar to be determined from the dispersion relation. Instead of a full replacement, a partial replacement of the dispersion terms are made so a form with better dispersion characteristics is obtained. Using $u_i = -g \nabla \eta$ for replacing the terms proportional to $\beta$ gives

$$u_i + (u \cdot \nabla) u_i + g \nabla \eta = \left(1 + \beta \frac{1}{2} \right) \nabla \cdot (h \bf{u})$$

$$+ \beta \frac{1}{6} \nabla \cdot (h \nabla \eta) - \left(1 + \beta \right) \frac{1}{6} \nabla \cdot (h \nabla \bf{u})$$

which is a momentum equation with mixed dispersion terms. Setting $\beta = 0$ recovers the original equation, while $\beta = -1$ corresponds to replacing $u_i$ with $-g \nabla \eta$ in equation (2). Equations (2) and (4) constitute the improved Boussinesq Equations.

2.2 Specification of dispersion parameter

Linearized 1-D Boussinesq Equations for mildly varying depth is formulated as follows. The continuity equation in expanded form

$$\frac{\partial \eta}{\partial t} + \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0$$

The momentum equation can be expanded as

$$(5)$$

$$\frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} = -h \frac{\partial}{\partial x} \frac{\partial u}{\partial x} + h \frac{\partial^3 u}{\partial x^3}$$

where $h \partial \eta / \partial x$ is the linear shoaling term while $h^2 \partial / \partial x$ is the linear dispersing term. Linearized 1-D Boussinesq Equations for constant depth simplify to the following equations.

$$(6)$$

$$(7)$$

Combining equations (6) and (7) by cross-differentiation the 1-D Boussinesq equations for constant depth is obtained as

$$(9)$$

where $h^2 \partial \eta / \partial x \partial t$ is the linear dispersion depth. Water waves of different wavelength travel with different phase speeds, a phenomenon known as frequency dispersion. For the case of infinitesimal wave amplitude, the terminology is linear frequency dispersion. The frequency dispersion characteristics of a Boussinesq-type of equation can be used to determine the range of wavelengths for which it is a valid approximation. Assume $\eta = \eta e^{i(kz + \omega t)}$ so that $\eta_x = -i \omega \eta e^{i(kz + \omega t)}$ and $\eta_{xx} = -i k^2 \eta e^{i(kz + \omega t)}$. Substituting these expressions into equation (9) gives

$$\omega^2 \left(1 + \frac{k^2 h^2}{3}\right) = g h k^2$$

which can be rewritten as,

$$\omega^2 = \frac{k^2}{1 + \frac{k^2 h^2}{3}} \times \frac{1 - \frac{k^2 h^2}{3}}{gh}$$
Figure 1. Dispersion curves for various values of dispersion parameter \( \beta \) compared with linear theory.

\[
c^2 = gh \left( 1 - \frac{k^2 h^2}{3} \right)
\]

Here \( k^2 h^2 / 3 \) shows the correction to the wave celerity due to the inclusion of the weak dispersion effect. Considering the improved Boussinesq equations, in linearized forms equations (2) and (4) yield the following dispersion relation evaluated by Beji and Nadaoka [5]:

\[
\frac{\omega^2}{gk} = \frac{kh(1 + \beta k^2 h^2 / 3)}{[1 + (1 + \beta) k^2 h^2 / 3]}
\]

Equation (13) is specified according to matching the resulting dispersion relation with a second order Padé expansion of the linear theory dispersion and \( \beta \) is determined from this second order Padé expansion of the linear theory dispersion relation \( \omega^2 / gk = \tan \theta k^2 \):

\[
\frac{\omega^2}{gk} = \frac{kh + k^2 h^2 / 15 + 2k^2 h^2 / 5}{1}
\]

In order that Equation (13) be identical with Equation (14) \( \beta \) should be set to 1/5. Figure 1 compares various values of dispersion parameters with the exact expression of linear theory. Among these asymptotic expansions, the one corresponding the Padé type expansion is the best. Thus, when \( \beta = 1/5 \), the model may propagate relatively shorter waves (\( \lambda = 1 \)) with acceptable errors in amplitude and celerity.

3 A NEW DISCRETIZATION SCHEME FOR 1-D IMPROVED BOUSSINESQ EQUATIONS

The finite difference method is the most natural way of solving a PDE directly in an approximate manner. The idea behind this is to discretize the continuous time and space into a finite number of discrete grid points and then to approximate the local derivatives at these grid points with finite difference schemes. For numerical modeling, the discretization of the variables \( u, v \) and \( \eta \) are necessary in order to solve momentum and continuity equations. Arakawa C grid which is shown in Figure 2, is the most appropriate system since it enables the discretization of the continuity equation in the most accurate manner. Here, \( u \) and \( \eta \) represent the horizontal velocity and the free surface displacement respectively. The surface displacement is obtained from an semi-explicit discretization of the continuity equation which is,

\[
\eta_t + \frac{\partial}{\partial x}[(h + \eta)u] = 0
\]

Multiplying both sides of the continuity equation by \( \Delta t \) and differentiating with respect to \( x \) gives:

\[
\left( \frac{\partial \eta}{\partial x} \right)_{i}^{k+1} = \left( \frac{\partial \eta}{\partial x} \right)_{i}^{k} - \frac{1}{2} h_{i+1/2} \left( \frac{\partial^2 u}{\partial x^2} \right)_{i}^{k} + \frac{1}{2} h_{i-1/2} \left( \frac{\partial^2 u}{\partial x^2} \right)_{i}^{k} \Delta t
\]

(16)

Discretization of the momentum equation is given as follows noting that all spatial derivatives are centered at the grid point where \( u_i^k \) is located.
where \( q = (3H/d)^{2/3}(x - Ct)/2d \). The free-surface elevation, particle velocities, and pressure may be expressed respectively as follows

\[
\eta = \frac{u}{\sqrt{gd\ D}}
\]

\[
u = \frac{H}{\rho g H} \Delta p
\]

\[
\Delta p = \text{sech}^2 q
\]

where \( \Delta p \) is the difference in pressure at a point under the wave due to the presence of the solitary wave. To second approximation, this pressure difference is given by

\[
\Delta p = \frac{3H}{4d} \left[ 1 - \left( \frac{Y_s}{d} \right)^2 \right]
\]

Substituting \( \partial \eta / \partial x \) from equation (16) into the discretized \( \chi \)-momentum equation (18) and multiplying by \( \Delta t \) gives the expression which is essentially a tridiagonal matrix system for \( u_{i+1}^k, u_i^{k+1} \) and \( u_{i-1}^k \).

4 TEST CASES FOR THE VERIFICATION OF THE BOUSSINESQ MODEL

4.1 An analytical solution of Boussinesq equations: Solitary waves

The most elementary analytical solution of Boussinesq equations is a solitary wave. A solitary wave is a wave with only crest and a surface profile lying entirely above the still water level. It is neither oscillatory nor does it exhibit a trough. The solitary wave can be defined as a wave of translation since the water particles are displaced at a distance in the direction of wave propagation as the wave passes. A true solitary wave cannot be formed in nature because there are usually small dispersive waves at the trailing edge of the wave. On the other hand, long waves such as tsunamis and waves resulting from large displacements of water caused by such phenomena as landslides and earthquakes sometimes behave approximately like solitary waves. Also, when an oscillatory wave moves into shallow water, it may often be approximated by a solitary wave. In this situation, the wave amplitude becomes progressively higher, the crests become shorter and more pointed, and the trough becomes longer and flatter. Only one parameter, wave steepness, \( \varepsilon = H/d \) is needed to specify a solitary wave because both wavelength and period of solitary waves are infinite. To the lowest order, the solitary wave profile varies as \( \text{sech}^2 q \),
Figure 3. Solitary waves for different wave heights when $\beta = 0$.

by calculating the relative error percentage for different $\epsilon$ values. It is observed that as nonlinearity parameter, $\epsilon$, increases, the relative error percentage increases linearly up to 13% for $\epsilon = 0.4$.

4.2 Comparison of analytical solution to linear shallow water wave equations

The linearized long-wave equations in 1-D in presence of a pressure term are given by,
\[ \eta_t + hu_x = 0 \quad (25) \]
\[ u_t + g \eta_x = -\frac{1}{\rho} VP \quad (26) \]

where \( \eta \) is free surface elevation, \( u \) horizontal velocity component, \( p \) surface pressure, \( h \) constant water depth and \( g \) gravitational acceleration. Due to the moving pressure, two different free surface waves moving in opposite directions are generated. The mathematical problem can be separated as the free surface wave problem and the pressure wave problem. Assume that the moving pressure is \( P = P_s F(x - Vt) \) and for the two free waves the profiles are \( \eta_1 = a_0 F(x - c_0 t) \) and \( \eta_2 = a_0 F(x + c_0 t) \). Besides these free waves the forced wave profile is \( \eta_3 = a_3 F(x - Vt) \) and the velocity of the forced wave is \( u_3 = b_3 F(x - Vt) \). For the right moving sinusoidal wave, wave surface profile can be described as,

\[ \eta = a_0 \sin(kx - \omega t) \quad (27) \]

where \( a_0 \) is the wave amplitude, \( k \) is the wave number. Differentiating the above equation with respect to \( x \) gives

\[ \eta_x = ka_0 \cos(kx - \omega t) \quad (28) \]

From the momentum equation (26) for the unforced free wave case (no pressure) \( u_t = -g \eta_x \).

Substituting \( \eta_x \) into this expression

\[ u_t = -gka_0 \cos(kx - \omega t) \quad (29) \]

Integrating over time, the horizontal velocity \( u \) is found as,

\[ u = \frac{gk}{\omega}a_0 \sin(kx - \omega t) \quad (30) \]

where \( a_0 \sin(kx - \omega t) \) represents \( \eta \) itself and \( \omega = kc \). Substituting these expressions, the horizontal velocity is found as, \( u = c\eta/\omega \). Therefore, right moving wave velocity is \( u_1 = c_0 \eta_1/\omega \) and the left moving wave velocity is \( u_2 = -c_0 \eta_2/\omega \). Now considering the forced wave case with pressure gradient and substituting \( \eta_3 \) into the continuity equation (25) by taking the time derivative. \( u_3 = Vb_3 F(x - Vt)/h \). Noting that \( u_t = b_3 F(x - Vt) \) gives

\[ b_3 = -\frac{P_s}{\rho \left( gh - V^2 \right)} \quad (32) \]

Substituting \( u_3 = b_3 F(x - Vt) \) and \( P = P_s F(x - Vt) \) into the momentum equation (26) by taking the derivative with respect to \( t \) and \( x \) respectively, \( u_3 = \rho g Va_3 F(x - Vt) + PF(x - Vt) / \rho \). Noting that \( u_t = b_3 F(x - Vt) \) gives

\[ b_3 = -\frac{P_s}{\rho \left( gh - V^2 \right)} \quad (32) \]

Substituting \( a_3 \) into the expression \( \eta_3 = a_3 b_3 F(x - Vt) \) results in \( \eta_1 = h \eta_3 P_s F(x - Vt) / \rho (gh - V^2) \). \( u_1 \) and \( u_2 \) are found by substituting \( \eta_1 \) and \( \eta_2 \) into the continuity equation (25) respectively. After these substitution \( u_1 = a_0 c_0 F(x - c_0 t)/h \) and \( u_2 = -a_0 c_0 F(x + c_0 t)/h \). The boundary conditions are,

\[ u_1 + u_2 + u_3 = 0 \quad (33) \]
\[ \eta_1 + \eta_2 + \eta_3 = 0 \quad (34) \]

Substituting free and forced solutions into (33) and (34) for \( t = 0 \) and solving for \( a_2 \) and \( a_3 \) gives

\[ a_2 \quad (35) \]
\[ a_3 \quad (36) \]

Finally, for three different wave profiles the following expressions for \( \eta \) are obtained.

\[ \eta_1 = \frac{h \eta_3 (c_0 + V)}{2 \rho \left( gh - V^2 \right)} F(x - c_0 t) \quad (37) \]
\[ \eta_2 = \frac{h \eta_3 (c_0 - V)}{2 \rho \left( gh - V^2 \right)} F(x + c_0 t) \quad (38) \]
\[ \eta_3 = -\frac{h \eta_3 V}{\rho \left( gh - V^2 \right)} F(x - Vt) \quad (39) \]

The corresponding velocities are computed likewise. Let’s assume that the moving pressure field is represented by \( F(x) = 1.25 \exp(-\chi/250) \) where \( \chi = x - Vt \). In this case we choose, \( h = 20 \text{ m} \), \( \rho_s = -4905 \text{ kg/m}^3 \), and \( \rho = 1000 \text{ kg/m}^3 \). The length of the computational domain is \( 20 \text{ m} \) and time step is \( 1 \text{ s} \). Solutions for linear shallow water waves at \( t = 50 \text{ and } 100 \text{ s} \) and for the velocities, \( V = 10 \text{ and } 18 \text{ m/s} \) are in Figure 6.

The same analytical solution is compared with one dimensional Boussinesq model when \( \beta = 1/5 \) for the velocities \( V = 10 \text{ and } 18 \text{ m/s} \) at \( t = 50 \text{ and } 100 \text{ s} \). As it can be seen from Figure 7, the analytical and numerical solutions are again in agreement.
In Figure 8 the average (using 500 points) error percentages between analytical and computational surface elevations are shown for five different pressure field speeds $V = 5, 10, 15, 20$ and $25$ m/s which corresponds to depth Froude numbers, $0.4, 0.7, 1.1, 1.4$ and $1.8$ at $t = 50$ s when $\beta = 1/5$. Results show that around Froude number 1, the relative error percentage takes its maximum value and as Froude number exceeds 1, the error percentage decreases.

Figure 6. Comparison of numerical and analytical solutions of linear shallow water waves generated by a moving pressure at $t = 50$ s and $t = 100$ s.

Figure 7. Comparison of analytical solution with linear shallow water waves and 1-D Boussinesq solution generated by a moving pressure when $\beta = 1/5$ at $t = 50$ s and $t = 100$ s.

Figure 8. Average relative error of the calculated and analytical surface elevation versus Froude number.