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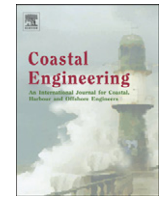

Short communication

Improved explicit approximation of linear dispersion relationship for gravity waves

S. Beji *

Faculty of Naval Architecture and Ocean Engineering, Istanbul Technical University, Maslak 34469, Istanbul, Turkey

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ABSTRACT

For water waves the transcendental dispersion relationship is solved by iterative methods when wave period and water depth are given and wavelength or wave number is required. A highly accurate explicit approximation to linear dispersion relationship is proposed based on Eckart's explicit relationship. While Eckart's expression is accurate to within 5%, the improved relationship has a maximum relative error of less than 0.05%. A simpler form of the relationship with 0.2% accuracy is also given.

1. Introduction

Water wave dispersion relationship is usually solved for the wave number for a given wave period and a water depth. The solution process always requires some kind of iterative approach. Before the availability of electronic calculators this problem was especially tiresome therefore Shore Protection Manual (Shore Protection Manual, 1984) was suggesting graphical methods besides some tables. Eckart (1952) was probably the first to introduce an explicit approximate dispersion relationship with accuracy of 5%. Afterwards several explicit approximations have been proposed; Hunt (1979) gave an accurate explicit polynomial solution by Pade's form; but his solution is rather complicated. Wu and Thornton (1986) developed two series solutions in a shallow water region and in a deep water region with overall accuracy of 0.05%. Fenton and McKee (1990) also gave an empirical explicit approximation which is accurate to within 1.5% over all relative wavelengths. More recently, Guo (2002) has presented a relatively simple formula with 0.75% accuracy. Here, an empirical correction is suggested to improve Eckart's formula to an accuracy of 0.05%. An even simpler expression with 0.2% accuracy is also given. The given formulas are uniformly valid over the entire range of relative depths and may be used easily for practical purposes.

2. Improvement of Eckart's formula

Among several forms of the exact[1] dispersion relationship consider the following expression givenin terms of the dispersion parameter $\mu = kh = 2\pi h/L$ and $\mu_0 = k_0 h = 2\pi h/L_0$ with k being the wave number, h the water depth, L the wavelength, and L_0 the deep water wavelength calculated from $L_0 = gT^2/2\pi$ for a given wave period T and gravitational acceleration g.

$$\mu = \mu_0 / \tanh \mu. \tag{1}$$

Obviously μ cannot be obtained explicitly and some sort of iterative scheme is necessary. On the other hand, the approximate explicit dispersion relationship of Eckart (1952) is

$$\mu = \mu_0 / \sqrt{\tanh \mu_0} \tag{2}$$

which gives μ at once for a given wave period and water depth. However, Eq. (2) has a maximum relative error of 5%, which is rather high. Here, an empirical correction is suggested to improve this formula to an accuracy of 0.05%.The resulting formula is uniformly valid over the entire range of relative water depths and suitable for practical use. An important point to note is that both the exact expression (1) and the approximation (2) converge to the same limits for very small and very large μ and μ_0 values:

$$\mu \rightarrow \sqrt{\mu_0} \text{ for } \mu,\ \mu_0 \ll 1 \qquad \text{and} \qquad \mu \rightarrow \mu_0 \text{ for } \mu,\ \mu_0 \gg 1. \tag{3}$$

The improved form of Eckart's dispersion relationship is defined as $\mu = \mu_a(1 + f_c)$ with μ_a being the approximation given by Eq.(2) as μ, and f_c is a suitable correction function. The form of f_c, which tends to zero for both $\mu_0 \rightarrow 0$ and $\mu_0 \rightarrow \infty$, is proposed as

$$f_c(\mu_0) = \mu_0^{\alpha} e^{-(\beta_0 + \beta_1 \mu_0 + \beta_2 \mu_0^2)} \tag{4}$$

where α, and β_0, β_1, β_2 are positive constants to be determined by minimizing the error between the numerical solution of the exact dispersion relation and the outcome of the improved approximation. The

* Tel.: +90 212 285 6442; fax: +90 212 285 6454.
E-mail address: sbeji@itu.edu.tr.

[1] Here, "exact" is used in the sense that the expression is exact to the second-order as formulated by linear theory or Stokes's second-order theory.

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numerical solution of the exact dispersion relationship for each μ_0 is obtained by applying the midpoint formula to successive iterations until the absolute value of the difference between two successive values is less than 10^{-6}. Employing the least-square method for the range $0.05 \le \mu_0 \le 2.45$ with 1000 points, and rounding off the resulting values for ease of use gives $\alpha = 1.09$, and $\beta_0 = 1.55$, $\beta_1 = 1.30$, $\beta_2 = 0.216$. The range for the application of the least-square method is selected to cover the region with relatively higher error percentage. The smaller $\mu_0 < 0.05$ and larger $\mu_0 > 2.45$ regions are purposely left out as the values obtained from the approximate relationship of Eckart become identical with the exact results while the correction function f_c vanishes in both limits as indicated above. Thus, the empirically improved relationship reads

$$\mu = \mu_0 \left[1 + \mu_0^{1.09} e^{-\left(1.55 + 1.30\mu_0 + 0.216\mu_0^2\right)}\right] \Big/ \sqrt{\tanh \mu_0} \qquad (5)$$

where $\mu_0 = k_0 h = 4\pi^2 h / gT^2$ as defined before and μ is the new improved approximate solution of the dispersion relationship with a maximum relative error of -0.044% for $\mu_0 = 0.3$, and going to zero as the relative water depth goes to zero or to infinity. A somewhat simpler form is possible by letting

$$f_c(\mu_0) = \mu_0^{\alpha} e^{-(\beta_0 + \beta_1 \mu_0)}. \qquad (6)$$

Determining α, and β_0, β_1 similarly gives $\alpha = 1.3$, $\beta_0 = 1.1$, $\beta_1 = 2.0$:

$$\mu = \mu_0 \left[1 + \mu_0^{1.3} e^{-(1.1 + 2.0\mu_0)}\right] \Big/ \sqrt{\tanh \mu_0} \qquad (7)$$

which has a maximum relative error of 0.187% for $\mu_0 = 0.06$ and less for the rest of the relative depths.

3. Conclusion

Eckart's explicit dispersion relationship is improved by introducing an empirical correction function. The resulting expression is valid for the entire range of relative water depths and accurate to within 0.05%. This error corresponds to millimeters in computing wavelengths of meters.

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