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A Systematic Approach to the Exact Roots of Polynomials

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Abstract. A unified framework is introduced for obtaining the exact roots of a polynomial by establishing a corresponding polynomial of one degree less. The approach gives the well-known solutions for the second and third degree polynomials and a new solution for the quartic equation, which is different in form from the classical Ferrari-Cardan solution. In accord with Abel's proof, the method produces no solution for the quintic equation.

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1. Introduction

Obtaining the roots of a polynomial is one of the oldest problems of algebra. While the solution of the second order equation was known quite earlier, it was not until Tartaglia, Cardan and Ferrari in the 1540's that the general solutions of the third and fourth order equations became known. In the 17th and 18th century innumerable futile attempts were made to solve the equation of the fifth degree. Finally in 1824 Abel, in his well-known work on the quintic equation, proved the impossibility of solving general equations of the fifth and higher degrees by means of radicals [1, pages 207-212, 261-266].

In this work a systematic approach is introduced for the exact solution of a degree n equation by first applying a Tschirnhaus transformation [2, pages 161-163] and then establishing a corresponding degree $(n-1)$ equation whose roots facilitate the solution sought. The approach is termed systematic in the sense that the same technique is applied to the second, third, fourth, and fifth degree equations by straightforward extensions without altering the established framework. The results for the second and third degree equations are in complete agreement with the known solutions. The solution obtained for the fourth degree equation is completely new and different in form from the solution of Cardan and Ferrari. Application of the technique to the quintic equation reveals that the solution is not possible.

2. Quadratic equation

In order to clarify the unified nature of the approach the simplest case of the second degree is considered first. Let a second degree polynomial be in the form

$$P_2(x) = x^2 + ax + b \tag{2.1}$$

with its roots x_1 and x_2. Introducing a new variable y and a constant $\theta = -a/2$ such that $x = y + \theta = y - a/2$ gives

$$Q_2(y) = (y - a/2)^2 + a(y - a/2) + b = y^2 + (-a^2 + 4b)/4. \tag{2.2}$$

Noting that $P_2(\theta) = (-a^2 + 4b)/4$ enables us to write

$$Q_2(y) = y^2 + \frac{P_2(\theta)}{0!} \tag{2.3}$$

where $\theta = -a/2$ is the constant employed in the Tschirnhaus transformation. When applied to an nth degree polynomial the Tschirnhaus transformation makes the coefficient of the $(n-1)$th degree term zero. The zero factorial appearing in (2.3) is not necessary; however, justification for its use shall be evident later. Note that a compact form for the transformed equation is introduced by evaluating the polynomial (and later its derivatives) for $\theta = -a/2$. Such a formulation is particularly advantageous for higher order polynomials.

Comparing now equation (2.1) with equation (2.3) we see that instead of two independent constants a and b appearing in (2.1) we have just one independent constant $P_2(\theta)$ in (2.3). Reduction in the number of independent constants implies that in the solution process just a single unknown quantity, say y_1, would be sufficient. The precise meaning of this statement shall be clear when the method is applied to higher degree equations. Designating the root of (2.3) by $y = y_1$ and solving for y_1 gives

$$y_1^2 + P_2(\theta) = 0, \qquad y_1 = \pm\sqrt{-P_2(\theta)}. \tag{2.4}$$

Two solutions of equation (2.1) are then

$$x_1 = \theta + \alpha_1\sqrt{-P_2(\theta)}, \qquad x_2 = \theta + \alpha_2\sqrt{-P_2(\theta)} \tag{2.5}$$

where $\alpha_1 = +1$ and $\alpha_2 = -1$ are the roots of $\alpha^2 = 1$.

3. Cubic equation

Following the approach used for the second degree equation we begin with the cubic polynomial

$$P_3(x) = x^3 + ax^2 + bx + c \tag{3.1}$$

whose roots are assumed to be x_1, x_2, and x_3. Defining now $\theta = -a/3$ and applying the change of variable $x = y + \theta = y - a/3$ as in §2 results in

$$\begin{aligned} Q_3(y) &= (y - a/3)^3 + a(y - a/3)^2 + b(y - a/3) + c \\ &= y^3 + [(-a^2 + 3b)/3]y + (2a^3 - 9ab + 27c)/27. \end{aligned} \tag{3.2}$$

Again we note that

$$\begin{aligned} P_2(\theta) &= [dP_3(x)/dx]_{x=\theta} = 3\theta^2 + 2a\theta + b = (-a^2 + 3b)/3 \\ P_3(\theta) &= \theta^3 + a\theta^2 + b\theta + c = (2a^3 - 9ab + 27c)/27 \end{aligned}$$

so that equation (3.2) may be written as

$$Q_3(y) = y^3 + \frac{P_2(\theta)}{1!}y + \frac{P_3(\theta)}{0!} \tag{3.3}$$

where $\theta = -a/3$ is the Tschirnhaus constant for a cubic polynomial.

Similar to the previous case, three independent constants, a, b, c, have been reduced to two new independent constants $P_2(\theta)$ and $P_3(\theta)$. Therefore, only two independent parameters y_1 and y_2 should be sufficient to solve equation (3.3). Since we are going to introduce only two parameters we expect to obtain a corresponding second order equation whose solution would facilitate the solution of the cubic equation.

Let $y = y_1 + y_2$ be the root of (3.3):

$$(y_1 + y_2)^3 + P_2(\theta)(y_1 + y_2) + P_3(\theta) = 0. \tag{3.4}$$

Re-arranging by factoring out $(y_1 + y_2)$, which is supposed to be non-zero, gives

$$[y_1^3 + y_2^3 + P_3(\theta)] + [3y_1y_2 + P_2(\theta)](y_1 + y_2) = 0, \tag{3.5}$$

which in turn requires

$$y_1^3 + y_2^3 = -P_3(\theta), \qquad y_1y_2 = -P_2(\theta)/3 \tag{3.6}$$

if (3.5) is to be satisfied. From (3.6) it is obvious that a second order polynomial whose roots are $Y_1 = y_1^3$ and $Y_2 = y_2^3$ may be established easily as

$$Y^2 + P_3(\theta)Y - P_2^3(\theta)/27 = 0. \tag{3.7}$$

The solutions of (3.7) are (see [3] for details regarding the following forms)

$$y_1 = Y_1^{1/3} = \sqrt[3]{-P_3(\theta)/2 + \sqrt{P_3^2(\theta)/4 + P_2^3(\theta)/27}}, \tag{3.8}$$

$$y_2 = Y_2^{1/3} = \sqrt[3]{-P_3(\theta)/2 - \sqrt{P_3^2(\theta)/4 + P_2^3(\theta)/27}}. \tag{3.9}$$

In evaluating the cubic roots in (3.8) and (3.9) it is necessary to consider all three possibilities $\alpha_1 = +1$, $\alpha_2 = (-1 + \mathrm{i}\sqrt{3})/2$, and $\alpha_3 = (-1 - \mathrm{i}\sqrt{3})/2$ for the roots of $\alpha^3 = +1$. Therefore, in principle $3^2 = 9$ combinations are possible for the arrangement of $y_1 + y_2$. However, equation (3.6) demands $y_1y_2 = -P_2(\theta)/3$ so that an acceptable combination must satisfy the condition $\alpha_i\alpha_j = +1$. Hence, combinations such as $\alpha_1\alpha_2$, $\alpha_2\alpha_1$, $\alpha_1\alpha_3$, $\alpha_3\alpha_1$, $\alpha_2\alpha_2$, and $\alpha_3\alpha_3$ (which give imaginary results) are all eliminated. Thus, the following three combinations, which satisfy $\alpha_1\alpha_1 = +1$, $\alpha_2\alpha_3 = +1$, $\alpha_3\alpha_2 = +1$, remain as valid solutions

$$\alpha_1 y_1 + \alpha_1 y_2, \qquad \alpha_2 y_1 + \alpha_3 y_2, \qquad \alpha_3 y_1 + \alpha_2 y_2$$

Hence the final solutions to the roots of cubic polynomial given by (3.1) are

$$\begin{aligned} x_1 &= \theta + \alpha_1 y_1 + \alpha_1 y_2 \\ x_2 &= \theta + \alpha_2 y_1 + \alpha_3 y_2 \\ x_3 &= \theta + \alpha_3 y_1 + \alpha_2 y_2 \end{aligned} \tag{3.10}$$

where $\theta = -a/3$ and $P_3(\theta)$, $P_2(\theta)$ appearing in y_1 and y_2 are the given polynomial and its first derivative evaluated for θ, respectively.

The solutions given in (3.10) together with (3.8) and (3.9) are in perfect agreement with the well-known solutions [4, pages 115-130]. See also reference [5, pages 261-310, 190-193] for a full account of the subject with historical remarks.

4. Quartic equation

In line with the approach used for the quadratic and cubic equations the exact roots of a fourth degree polynomial are now formulated. The present solution is novel in the sense that it is not in the form of the classic solution of Ferrari and Cardan. Let a fourth order polynomial be of the form

$$P_4(x) = x^4 + ax^3 + bx^2 + cx + d \tag{4.1}$$

whose roots are assumed to be x_1, x_2, x_3, and x_4. Letting $\theta = -a/4$ and applying the change of variable $x = y + \theta = y - a/4$ as in §2 and §3 results in

$$\begin{aligned} Q_4(y) &= (y - a/4)^4 + a(y - a/4)^3 + b(y - a/4)^2 + c(y - a/4) + d \\ &= y^4 + [(-3a^2 + 8b)/8]y^2 + [(a^3 - 4ab + 8c)/8]y \\ &\qquad + (-3a^4 + 16a^2 b - 64ac + 256d)/256. \end{aligned} \tag{4.2}$$

We note the relations

$$\begin{aligned} P_2(\theta) &= \left[d^2 P_4(x)/dx^2\right]_{x=\theta} = 12\theta^2 + 6a\theta + 2b = (-3a^2 + 8b)/4, \\ P_3(\theta) &= \left[dP_4(x)/dx\right]_{x=\theta} = 4\theta^3 + 3a\theta^2 + 2b\theta + c = (a^3 - 4ab + 8c)/8, \\ P_4(\theta) &= \theta^4 + a\theta^3 + b\theta^2 + c\theta + d = (-3a^4 + 16a^2 b - 64ac + 256d)/256. \end{aligned}$$

Then, equation (4.2) may be written as

$$Q_4(y) = y^4 + \frac{P_2(\theta)}{2!}y^2 + \frac{P_3(\theta)}{1!}y + \frac{P_4(\theta)}{0!} \tag{4.3}$$

where $\theta = -a/4$ as indicated before. It is worth to remark again that four independent coefficients, a, b, c, d, have been reduced to three now: $P_2(\theta)$, $P_3(\theta)$, and $P_4(\theta)$. We may therefore attempt to solve equation (4.3) in terms of only three independent parameters, say y_1, y_2, y_3. As we are going to introduce three parameters we expect to obtain a corresponding third order equation whose solution would facilitate the solution of the fourth order equation.

Let $y = y_1 + y_2 + y_3$ be the root of (4.3):

$$(y_1 + y_2 + y_3)^4 + \frac{P_2(\theta)}{2}(y_1 + y_2 + y_3)^2 + P_3(\theta)(y_1 + y_2 + y_3) + P_4(\theta) = 0. \tag{4.4}$$

Expanding the second and fourth powers and gathering the terms proportional to $(y_1 + y_2 + y_3)$ and $(y_1y_2 + y_1y_3 + y_2y_3)$ gives

$$\begin{aligned} y_1^4 + y_2^4 + y_3^4 &+ 6(y_1^2y_2^2 + y_1^2y_3^2 + y_2^2y_3^2) + \frac{P_2(\theta)}{2}(y_1^2 + y_2^2 + y_3^2) + P_4(\theta) \\ &+ [4(y_1^2 + y_2^2 + y_3^2) + P_2(\theta)](y_1y_2 + y_1y_3 + y_2y_3) \\ &+ [8y_1y_2y_3 + P_3(\theta)](y_1 + y_2 + y_3) = 0. \end{aligned} \tag{4.5}$$

We first require that the terms inside the square brackets vanish separately so that

$$y_1^2 + y_2^2 + y_3^2 = -P_2(\theta)/4, \tag{4.6}$$

$$y_1y_2y_3 = -P_3(\theta)/8. \tag{4.7}$$

The next step is to equate the remaining terms, which appear in the first line of (4.5), to zero. Before doing so we square equation (4.6) and solve for $y_1^2y_2^2 + y_1^2y_3^2 + y_2^2y_3^2$ as

$$y_1^2y_2^2 + y_1^2y_3^2 + y_2^2y_3^2 = P_2^2(\theta)/32 - (y_1^4 + y_2^4 + y_3^4)/2 \tag{4.8}$$

which, together with (4.6), may be used in the remaining part of (4.5) to get

$$y_1^4 + y_2^4 + y_3^4 = P_2^2(\theta)/32 + P_4(\theta)/2. \tag{4.9}$$

Thus, as long as equations (4.6), (4.7), and (4.9) hold, equation (4.5) hence (4.4) is satisfied. However, for solving y_1, y_2, y_3 we still need to establish a corresponding cubic equation.

Using (4.9) in (4.8) now gives

$$y_1^2y_2^2 + y_1^2y_3^2 + y_2^2y_3^2 = P_2^2(\theta)/64 - P_4(\theta)/4 \tag{4.10}$$

which, after squaring, may be arranged as

$$y_1^4y_2^4 + y_1^4y_3^4 + y_2^4y_3^4 + 2y_1^2y_2^2y_3^2(y_1^2 + y_2^2 + y_3^2) = [P_2^2(\theta)/64 - P_4(\theta)/4]^2. \tag{4.11}$$

Making use of (4.6) and (4.7) in (4.11) results in

$$y_1^4y_2^4 + y_1^4y_3^4 + y_2^4y_3^4 = [P_2^2(\theta)/64 - P_4(\theta)/4]^2 + P_2(\theta)P_3^2(\theta)/128. \tag{4.12}$$

Equations (4.9), (4.12) and the fourth power of (4.7) now provide all the necessary equations for establishing a cubic polynomial whose roots are y_1^4, y_2^4, y_3^4:

$$\begin{aligned} Y^3 - [P_2^2(\theta)/32 + P_4(\theta)/2]Y^2 + \{[P_2^2(\theta)/64 - P_4(\theta)/4]^2 \\ +P_2(\theta)P_3^2(\theta)/128\}Y - [P_3^4(\theta)/4096] &= 0. \end{aligned} \tag{4.13}$$

The problem of obtaining the roots of a fourth order polynomial has thus been reduced to the problem of obtaining the roots of a cubic polynomial, which in turn may be reduced to solving a second degree equation.

Since $y_1 = \sqrt[4]{Y_1}$, etc. the solutions of $\alpha^4 = +1$ must be considered in establishing the appropriate combinations. In this case $\alpha_1 = +1$, $\alpha_2 = -1$, $\alpha_3 = +\mathrm{i}$, and $\alpha_4 = -\mathrm{i}$. By inspections similar but much lengthier than §3 the following four combinations are established as valid solutions to equation (4.3):

$$\begin{array}{ll} \alpha_1 y_1 + \alpha_1 y_2 + \alpha_3 y_3, & \alpha_1 y_1 + \alpha_2 y_2 + \alpha_4 y_3 \\ \alpha_2 y_1 + \alpha_1 y_2 + \alpha_4 y_3, & \alpha_2 y_1 + \alpha_2 y_2 + \alpha_3 y_3 \end{array}$$

An important side condition is that the real root must be assigned to y_1 unless all the roots are real with a zero root; in the latter case zero must be assigned to y_3. The final solutions to the roots of quartic polynomial given by (4.1) are

$$\begin{aligned} x_1 &= \theta + \alpha_1 y_1 + \alpha_1 y_2 + \alpha_3 y_3 \\ x_2 &= \theta + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_4 y_3 \\ x_3 &= \theta + \alpha_2 y_1 + \alpha_1 y_2 + \alpha_4 y_3 \\ x_4 &= \theta + \alpha_2 y_1 + \alpha_2 y_2 + \alpha_3 y_3 \end{aligned} \tag{4.14}$$

where $\theta = -a/4$ as defined before. It is understood that the fourth roots of Y_1, Y_2, and Y_3, which are necessary for obtaining $y_1 = \sqrt[4]{Y_1}$, etc. are computed as the first quadrant values.

In the development of these solutions the most challenging part is obviously to express $(y_1^4 y_2^4 + y_1^4 y_3^4 + y_2^4 y_3^4)$ using the available quantities $(y_1^2 + y_2^2 + y_3^2)$, $(y_1 y_2 y_3)$, and $(y_1^4 + y_2^4 + y_3^4)$. Above this is accomplished in an intuitive way; however, for fifth and higher order equations the intuitive approach is virtually impossible due to extremely large number of possibilities. A routinely applicable method is desirable. This is possible by dimensional analysis. Supposing each one of the quantities y_1, y_2, y_3 has the dimension of unity then the given quantities $(y_1^2 + y_2^2 + y_3^2)$, $(y_1 y_2 y_3)$, and $(y_1^4 + y_2^4 + y_3^4)$ would have the dimensions of D_2, D_3 and D_4, respectively. Since the aimed expression $(y_1^4 y_2^4 + y_1^4 y_3^4 + y_2^4 y_3^4)$ has the dimension of D_8, the possibilities of constructing a quantity of dimension D_8 using the quantities of dimensions D_2, D_3 and D_4 may be enumerated as $D_2 D_3^2$, $D_2^2 D_4$, D_2^4, and D_4^2. In other words, only these combinations would be dimensionally acceptable. If each possibility is multiplied by an unknown constant and all are added together, the constants may be determined by equating the resulting expression to the desired quantity:

$$\begin{aligned} &A(y_1^2 + y_2^2 + y_3^2)(y_1 y_2 y_3)^2 + B(y_1^2 + y_2^2 + y_3^2)^2(y_1^4 + y_2^4 + y_3^4) \\ &+C(y_1^2 + y_2^2 + y_3^2)^4 + D(y_1^4 + y_2^4 + y_3^4)^2 = y_1^4 y_2^4 + y_1^4 y_3^4 + y_2^4 y_3^4 \end{aligned} \tag{4.15}$$

where A, B, C, and D are the arbitrary constants to be determined. After expanding and re-arranging,

$$\begin{aligned} &(B + C + D)(y_1^8 + y_2^8 + y_3^8) + (2B + 6C + 2D)(y_1^4 y_2^4 + y_1^4 y_3^4 + y_2^4 y_3^4) \\ &\quad +(2B + 4C)(y_1^6 y_2^2 + y_1^2 y_2^6 + y_1^6 y_3^2 + y_2^6 y_3^2 + y_1^2 y_3^6 + y_2^2 y_3^6) \\ &+(A + 2B + 12C)(y_1^2 + y_2^2 + y_3^2)(y_1^2 y_2^2 y_3^2) = y_1^4 y_2^4 + y_1^4 y_3^4 + y_2^4 y_3^4 \end{aligned}$$

which in turn dictates the following equalities

$$\begin{array}{ll} B + C + D = 0, & 2B + 6C + 2D = 1, \\ 2B + 4C = 0, & A + 2B + 12C = 0. \end{array}$$

The solution is possible and may be easily obtained as

$$A = -2, \qquad B = -1/2, \qquad C = 1/4, \qquad D = 1/4.$$

Using these values in (4.15) it is at once possible to write $y_1^4 y_2^4 + y_1^4 y_3^4 + y_2^4 y_3^4$ in terms of the known quantities:

$$\begin{aligned} -2\left[-P_2(\theta)/4\right]\left[-P_3(\theta)/8\right]^2 - (1/2)\left[P_2^2(\theta)/32 + P_4(\theta)/2\right]\left[-P_2(\theta)/4\right]^2 \\ +(1/4)\left[-P_2(\theta)/4\right]^4 + (1/4)\left[P_2^2(\theta)/32 + P_4(\theta)/2\right]^2 \\ = y_1^4 y_2^4 + y_1^4 y_3^4 + y_2^4 y_3^4 \end{aligned}$$

or

$$y_1^4 y_2^4 + y_1^4 y_3^4 + y_2^4 y_3^4 = [P_2^2(\theta)/64 - P_4(\theta)/4]^2 + P_2(\theta)P_3^2(\theta)/128$$

as obtained previously in equation (4.12). We have thus established a routine approach which may be adapted for more complicated cases, such as a quintic equation.

4.1. Numerical example

The solution obtained for the quartic equation is new; therefore, a simple numerical demonstration may be useful. We begin with a polynomial with known roots so that the solution may be compared at once. Let

$$\begin{aligned} P_4(x) &= (x-1)(x+2)(x-1-\mathrm{i})(x-1+\mathrm{i}) \\ &= x^4 - x^3 - 2x^2 + 6x - 4 \end{aligned}$$

whose roots are obviously $x_1 = +1$, $x_2 = -2$, $x_3 = 1 + \mathrm{i}$, and $x_4 = 1 - \mathrm{i}$. The Tschirnhaus constant is $\theta = -a/4 = 1/4$ so that $P_2(1/4) = -19/4$, $P_3(1/4) = 39/8$, and $P_4(1/4) = -675/256$. We can make use of (4.3) to write down the transformed equation after the change of variable $x = y + \theta = y + 1/4$ as

$$Q_4(y) = y^4 - \frac{19}{8}y^2 + \frac{39}{8}y - \frac{675}{256}.$$

According to (4.13) the corresponding third order equation is

$$Y^3 + \frac{157}{256}Y^2 + \frac{9283}{256^2}Y - \frac{39^4}{256^3} = 0$$

which has the solutions (as can be computed from (3.8), (3.9), and (3.10)):

$$Y_1 = \frac{81}{256}, \qquad Y_2 = -\frac{119}{256} + \frac{120}{256}\,\mathrm{i}, \qquad Y_3 = -\frac{119}{256} - \frac{120}{256}\,\mathrm{i}.$$

Computing the fourth roots gives (using the first quadrant values)

$$y_1 = (Y_1)^{1/4} = \frac{3}{4}, \qquad y_2 = (Y_2)^{1/4} = \frac{3}{4} + \frac{1}{2}\,\mathrm{i}, \qquad y_3 = (Y_3)^{1/4} = \frac{1}{2} + \frac{3}{4}\,\mathrm{i}.$$

Following (4.14) we can compute the roots of the given equation as

$$\begin{aligned}
x_1 &= \frac{1}{4} + 1\cdot\left(\frac{3}{4}\right) + 1\cdot\left(\frac{3}{4}+\frac{1}{2}\mathrm{i}\right) + \mathrm{i}\cdot\left(\frac{1}{2}+\frac{3}{4}\mathrm{i}\right) = 1+\mathrm{i}\\
x_2 &= \frac{1}{4} + 1\cdot\left(\frac{3}{4}\right) - 1\cdot\left(\frac{3}{4}+\frac{1}{2}\mathrm{i}\right) - \mathrm{i}\cdot\left(\frac{1}{2}+\frac{3}{4}\mathrm{i}\right) = 1-\mathrm{i}\\
x_3 &= \frac{1}{4} - 1\cdot\left(\frac{3}{4}\right) + 1\cdot\left(\frac{3}{4}+\frac{1}{2}\mathrm{i}\right) - \mathrm{i}\cdot\left(\frac{1}{2}+\frac{3}{4}\mathrm{i}\right) = 1\\
x_4 &= \frac{1}{4} - 1\cdot\left(\frac{3}{4}\right) - 1\cdot\left(\frac{3}{4}+12\mathrm{i}\right) + \mathrm{i}\cdot\left(\frac{1}{2}+\frac{3}{4}\mathrm{i}\right) = -2
\end{aligned}$$

which are the expected values. Note that the real root 3/4 has been assigned to y_1 according to the condition stated in §4.

5. Quintic equation

Abel proved the impossibility of formulating the roots of the fifth and higher order polynomials by means of radicals. Here, we shall apply the method used in this work to the quintic equation and likewise arrive at the conclusion that the solution is not possible. Let a fifth order polynomial be of the form

$$P_5(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e \tag{5.1}$$

and introduce $\theta = -a/5$ such that $x = y + \theta = y - a/5$ as in the previous cases. The transformed polynomial becomes

$$Q_5(y) = y^5 + \frac{P_2(\theta)}{3!}y^3 + \frac{P_3(\theta)}{2!}y^2 + \frac{P_4(\theta)}{1!}y + \frac{P_5(\theta)}{0!} \tag{5.2}$$

where

$$\begin{aligned}
P_2(\theta) &= \left[d^3P_5(x)/dx^3\right]_{x=\theta} = (-12a^2 + 30b)/5\\
P_3(\theta) &= \left[d^2P_5(x)/dx^2\right]_{x=\theta} = (8a^3 - 30ab + 50c)/25\\
P_4(\theta) &= \left[dP_5(x)/dx\right]_{x=\theta} = (-3a^4 + 15a^2b - 50ac + 125d)/125\\
P_5(\theta) &= (4a^5 - 25a^3b + 125a^2c - 625ad + 3125e)/3125
\end{aligned}$$

Following the established approach let $y = y_1 + y_2 + y_3 + y_4$ be the root of (5.2)

$$\begin{aligned}
(y_1+y_2+y_3+y_4)^5 &+ \frac{P_2(\theta)}{6}(y_1+y_2+y_3+y_4)^3\\
+\frac{P_3(\theta)}{2}(y_1+y_2+y_3+y_4)^2 &+ P_4(\theta)(y_1+y_2+y_3+y_4) + P_5(\theta) = 0.
\end{aligned}$$

Expanding the second, third, and fifth powers, manipulating and gathering the terms proportional to $(y_1+y_2+y_3+y_4)$, $(y_1y_2+y_1y_3+y_2y_3+y_1y_4+y_2y_4+y_3y_4)$, and $(y_1y_2y_3 + y_1y_2y_4 + y_1y_3y_4 + y_2y_3y_4)$ gives

$$\begin{aligned}
6(y_1^5 + y_2^5 + y_3^5 + y_4^5) - 20(y_1^3 + y_2^3 + y_3^3 + y_4^3)(y_1^2 + y_2^2 + y_3^2 + y_4^2) \\
-P_2(\theta)(y_1^3 + y_2^3 + y_3^3 + y_4^3)/3 + P_3(\theta)(y_1^2 + y_2^2 + y_3^2 + y_4^2)/2 + P_5(\theta) \\
+[15(y_1^2 + y_2^2 + y_3^2 + y_4^2)^2 + P_2(\theta)(y_1^2 + y_2^2 + y_3^2 + y_4^2)/2 \\
+30(y_1y_2y_3y_4) + P_4(\theta)](y_1 + y_2 + y_3 + y_4) \\
-[10(y_1^3 + y_2^3 + y_3^3 + y_4^3) - P_3(\theta)](y_1y_2 + y_1y_3 + y_2y_3 + y_1y_4 + y_2y_4 + y_3y_4) \\
+[30(y_1^2 + y_2^2 + y_3^2 + y_4^2) + P_2(\theta)](y_1y_2y_3 + y_1y_2y_4 + y_1y_3y_4 + y_2y_3y_4) = 0.
\end{aligned}$$

First requiring that the terms inside the brackets vanish separately and then using the expressions obtained in the remaining parts of the equation provides the following equalities

$$y_1^5 + y_2^5 + y_3^5 + y_4^5 = -\frac{1}{6}\left[\frac{P_2(\theta)P_3(\theta)}{60} + P_5(\theta)\right], \qquad y_1y_2y_3y_4 = -\frac{P_4(\theta)}{30},$$
$$y_1^3 + y_2^3 + y_3^3 + y_4^3 = \frac{P_3(\theta)}{10}, \qquad y_1^2 + y_2^2 + y_3^2 + y_4^2 = -\frac{P_2(\theta)}{30}.$$

Obviously now the quantities $(y_1^5y_2^5 + y_1^5y_3^5 + y_2^5y_3^5 + y_1^5y_4^5 + y_2^5y_4^5 + y_3^5y_4^5)$, $(y_1^5y_2^5y_3^5 + y_1^5y_2^5y_4^5 + y_1^5y_3^5y_4^5 + y_2^5y_3^5y_4^5)$, and $(y_1^5y_2^5y_3^5y_4^5)$ are needed to establish a fourth-order equation for obtaining y_1^5,y_2^5,y_3^5,y_4^5 hence y_1,y_2,y_3,y_4. The quantity $(y_1^5y_2^5y_3^5y_4^5)$ is readily available by taking the fifth power of $y_1y_2y_3y_4 = -P_4(\theta)/30$; however, the others pose a real challenge. As indicated in §4 the intuitive approach is virtually impossible to adopt due to the extremely large number of possibilities. Therefore we resort to the dimensional approach and again suppose that each one of y_1, y_2, y_3, y_4 has the dimension of unity so that the available quantities $(y_1^2+y_2^2+y_3^2+y_4^2)$, $(y_1^3+y_2^3+y_3^3+y_4^3)$, $(y_1y_2y_3y_4)$, and $(y_1^5+y_2^5+y_3^5+y_4^5)$ would have the dimensions of D_2, D_3, D_4 and D_5, respectively. The sought expressions $(y_1^5y_2^5+y_1^5y_3^5+y_2^5y_3^5+y_1^5y_4^5+y_2^5y_4^5+y_3^5y_4^5)$ and $(y_1^5y_2^5y_3^5+y_1^5y_2^5y_4^5+y_1^5y_3^5y_4^5+y_2^5y_3^5y_4^5)$ have the dimensions of D_{10} and D_{15} in the given order. We may begin with the first expression and attempt to construct a quantity of dimension D_{10} using the quantities of dimensions D_2, D_3, D_4 and D_5. The possible combinations are D_2^5, $D_2^2D_3^2$, $D_2^3D_4$, $D_2D_4^2$, $D_2D_3D_5$, $D_3^2D_4$, D_5^2. Using each dimensionally acceptable combination we can construct a sum that would contain all these possibilities. Accordingly, each acceptable combination is multiplied by an unknown constant and all are added together. By equating the final expression to the aimed expression the unknown constants are determined:

$$\begin{aligned}
A(y_1^2 + y_2^2 + y_3^2 + y_4^2)^5 + B(y_1^2 + y_2^2 + y_3^2 + y_4^2)^2(y_1^3 + y_2^3 + y_3^3 + y_4^3)^2 \\
+C(y_1^2 + y_2^2 + y_3^2 + y_4^2)^3(y_1y_2y_3y_4) + D(y_1^2 + y_2^2 + y_3^2 + y_4^2)(y_1y_2y_3y_4)^2 \\
+E(y_1^2 + y_2^2 + y_3^2 + y_4^2)(y_1^3 + y_2^3 + y_3^3 + y_4^3)(y_1^5 + y_2^5 + y_3^5 + y_4^5) \\
+F(y_1^3 + y_2^3 + y_3^3 + y_4^3)^2(y_1y_2y_3y_4) + G(y_1^5 + y_2^5 + y_3^5 + y_4^5)^2 \\
= (y_1^5y_2^5 + y_1^5y_3^5 + y_2^5y_3^5 + y_1^5y_4^5 + y_2^5y_4^5 + y_3^5y_4^5)
\end{aligned}$$

After collecting the like terms the resulting set of linear algebraic equations is found to be *over-determined* hence it is concluded impossible to express $(y_1^5y_2^5 +$

$y_1^5y_3^5+y_2^5y_3^5+y_1^5y_4^5+y_2^5y_4^5+y_3^5y_4^5$) in terms of the known quantities $(y_1^2+y_2^2+y_3^2+y_4^2)$, $(y_1^3+y_2^3+y_3^3+y_4^3)$, $(y_1y_2y_3y_4)$, and $(y_1^5+y_2^5+y_3^5+y_4^5)$.

A preliminary attempt of expressing $(y_1^5y_2^5y_3^5+y_1^5y_2^5y_4^5+y_1^5y_3^5y_4^5+y_2^5y_3^5y_4^5)$ in terms of available quantities yields 14 possible combinations ($D_2^6D_3$, $D_2^5D_5$, $D_2^4D_3D_4$, $D_2^3D_3^3$, $D_2^3D_4D_5$, $D_2^2D_3^2D_5$, $D_2^2D_3D_4^2$, $D_2D_3^3D_4$, $D_2D_3D_5^2$, $D_2D_4^2D_5$, D_3^5, $D_3^2D_4D_5$, $D_3D_4^3$, D_5^3) and consequently much more complicated expressions. Since we have already failed to express one of the needed quantities it is deemed futile to pursue the matter further for the second quantity.

6. Concluding remarks

A well-defined technique for obtaining the roots of a polynomial of degree $n < 5$ is established. First an appropriate Tschirnhaus transformation is applied and then the coefficients of the transformed equation are expressed in convenient forms through the use of the polynomial and its derivatives evaluated for the Tschirnhaus constant. Finally, the solution of the transformed equation is sought as a summation of $n-1$ independent parameters which are to be arranged as the roots of a polynomial of degree $n-1$. The approach presented here may be viewed as a process of reducing the degree of a polynomial equation by one. This systematic methodology produces solutions for the second, third, and fourth order polynomials in exactly the same manner. Application of the method to the quintic equation reveals Abel's well-known conclusion that the problem is insolvable.

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