A time-dependent nonlinear mild-slope equation for water waves

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A weakly nonlinear and dispersive water wave equation, which in linearized form yields a new version of the time-dependent mild-slope equation of Smith & Sprinks (1975), is derived. The applicable spectral width of the new wave equation for random waves is found to be more satisfactory than that of Smith and Sprinks (1975). For very shallow depths the equation reduces to the combined form of Airy's nonlinear non-dispersive wave equations; if the lowest-order dispersion is retained it produces the combined form of Boussinesq's equations. In the deep-water limit the equation admits the second-order Stokes waves as analytical solutions. Furthermore, by introducing a right-moving coordinate transformation, the equation is recast into a unidirectional form, rendering the KdV equation in one limit while reproducing the second-order Stokes waves in the other.

1. Introduction

Since its first introduction more than two decades ago the mild-slope equation of Berkhoff (1972), or in more general time-dependent form Smith & Sprinks's (1975) equation, has gained a justified popularity in modelling coastal wave phenomenon, specifically the combined refraction and diffraction effects. During this period, on the one hand, further theoretical development of this equation, and, on the other hand, efficient numerical modelling of its various forms (see Panchang *et al.* 1991) have accumulated a considerable literature. Radder (1979) introduced the so-called parabolic approximation to Berkhoff's equation for computational efficiency. Liu & Tsay (1983) improved on Radder's work to account for the back scattered waves. An extended version of the time-dependent mild-slope equation which included the effects of rapidly varying depth undulations was derived by Kirby (1986). Chamberlain & Porter (1995) developed a modified mild-slope equation which contained Berkhoff's and Kirby's equations as special cases. Recently, Porter & Staziker (1995) have derived a jump condition for a bed profile with discontinuous slope and extended Chamberlain & Porter's modified mild-slope equation to a higher-order approximation.

As the role of nonlinear effects received increasing appreciation, inclusion of nonlinearity in wave models became a desirable prospect. Kirby & Dalrymple (1983) introduced a parabolic model with cubic nonlinearity, which was a general form of the equation given by Yue & Mei (1980). For weakly nonlinear Stokes-type waves Liu

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& Tsay (1984) developed a model in the form of a nonlinear Schrödinger equation with variable coefficients. Nonlinear refraction-diffraction of shallow water waves was treated by Liu *et al.* (1985) using the Boussinesq equations and the KP (Kadomtsev & Petviashvili) equation in connection with the parabolic equation method.

The aim of the present work is to combine the single-term forms of quite general fully dispersive weakly nonlinear wave equations introduced by Nadaoka *et al.* (1997). The linearized form of the combined equation can be put into the time-dependent form of the mild-slope equation and thus in its full form it may suitably be called a *time-dependent nonlinear mild-slope equation*. For incident wave frequencies different from the prescribed wave frequency of the model the linearized form of the proposed equation shows better dispersion characteristics than Smith & Sprinks's (1975) equation does, as is demonstrated here.

The following section states the continuity and momentum equations constituting the single-term forms of the wave equations by Nadaoka *et al.* (1997) and combines these equations into a single one. In § 3, Smith & Sprinks's (1975) time-dependent mild-slope equation is recovered from the linearized form of the new equation and the dispersion characteristics of these equations are compared. The degenerate forms of the new wave equation for shallow and deep water are worked out in § 4. Section 5 introduces a coordinate transformation to derive a one-way propagation model, which is, from the computational point of view, very attractive. Section 6 examines the degenerate forms of this equation, showing in particular that it corresponds to the KdV equation for shallow depths and admits the second-order Stokes waves as analytical solutions in deep water. In § 7 the performance of this unidirectional one-way wave equation is demonstrated by various numerical simulations. The last section is devoted to concluding remarks.

2. A time-dependent nonlinear mild-slope equation

Quite recently, Nadaoka *et al.* (1997) introduced a set of fully dispersive weakly nonlinear wave equations describing wave transformations over varying depth. These equations in their most general form are composed of several depth-dependence functions, each contributing to the dispersivity of the full set. More importantly, the applicable range of the single-component model (i.e. the wave equations derived from a single depth-dependence function) is not confined to the prescribed wave-number itself but to a narrow band of wave-numbers centred around this particular wave-number, as in the time-dependent mild-slope equation. This enhancement in dispersivity is brought about by the application of the Galerkin procedure, which may also be interpreted as the solvability condition invoked in derivation of slowly modulated wave envelope equations, such as the nonlinear Schrödinger equation (see for instance Mei 1983, p.611).

Considering the fact that a narrow-banded sea state centred around a certain dominant wave frequency may be described with sufficient accuracy by the single-component model equations it becomes a justifiable action to pay further attention to these equations in order that they may be put into convenient forms to render further theoretical and practical aspects. Nadaoka *et al.* (1997) give the following continuity and momentum equations as the single-component wave model, correct to the second order in nonlinearity:

$$\eta_t + \nabla \cdot \left[\left(\frac{C_p^2}{g} + \eta \right) \boldsymbol{u} \right] = 0, \tag{2.1}$$

$$C_pC_g\boldsymbol{u}_t + C_p^2\nabla\left[g\eta + \eta w_t + \tfrac{1}{2}\left(\boldsymbol{u}\cdot\boldsymbol{u} + w^2\right)\right] = \frac{C_p(C_p-C_g)}{k^2}\nabla(\nabla\cdot\boldsymbol{u}_t) + \nabla\left[\frac{C_p(C_p-C_g)}{k^2}\right](\nabla\cdot\boldsymbol{u}_t), \tag{2.2}$$

where η is the free surface displacement, $\boldsymbol{u}(u,v)$ the two-dimensional horizontal velocity vector and w the vertical component of the velocity both at the still water level $z=0$. C_p, C_g and k are, respectively, the phase and group velocities, and wavenumber computed according to linear theory for a prescribed dominant frequency ω and a local depth h. g is the gravitational acceleration, ∇ stands for the horizontal gradient operator with components $(\partial/\partial x, \partial/\partial y)$, and subscript t indicates partial differentiation with respect to time. Note that (2.1) and (2.2) are formulated for varying depth and therefore C_p, C_g and k are in general spatially varying quantities.

The objective here is to combine (2.1) and (2.2) into a single nonlinear wave equation describing the evolution of η in time and space. A perturbation procedure is used, which in essence serves as a simple tool to replace the nonlinear and linear shoaling terms with their approximate equivalents as obtained from the linearized equations. In this procedure a more general nonlinearity parameter ga/C_p^2 (Beji 1995) may be used as the small expansion term instead of one of the classical definitions ka or a/h. The parameter ga/C_p^2 embodies both ka and a/h as its special cases respectively for $C_p^2 = g/k$ (deep water) and $C_p^2 = gh$ (shallow water) and there remains no ambiguity about its applicable range. This important feature accords well with (2.1) and (2.2) which are valid at arbitrary depths. Actually, a suitable non-dimensionalization of (2.1) and (2.2) also yields the same nonlinearity parameter. For the present purposes however such a small non-dimensional parameter may be regarded as a bookkeeping device indicating the relative significance of the terms and therefore, for notational convenience, it may be used to label not only the nonlinear terms but also the linear shoaling terms (i.e. the terms proportional to the gradients of C_p, C_g and k). Thus, equations (2.1) and (2.2) are rewritten as

$$\eta_t + \frac{C_p^2}{g}(\nabla\cdot\boldsymbol{u}) + \varepsilon\nabla\left(\frac{C_p^2}{g}\right)\cdot\boldsymbol{u} + \varepsilon\nabla\cdot(\eta\boldsymbol{u}) = O(\varepsilon^2), \tag{2.3}$$

$$C_pC_g\boldsymbol{u}_t + C_p^2\nabla\left[g\eta + \varepsilon\eta w_t + \tfrac{1}{2}\varepsilon\left(\boldsymbol{u}\cdot\boldsymbol{u} + w^2\right)\right] - \frac{C_p(C_p-C_g)}{k^2}\nabla(\nabla\cdot\boldsymbol{u}_t) - \varepsilon\nabla\left[\frac{C_p(C_p-C_g)}{k^2}\right](\nabla\cdot\boldsymbol{u}_t) = O(\varepsilon^2), \tag{2.4}$$

where ε is a small non-dimensional parameter labelling the nonlinear and linear shoaling terms.

Let us introduce the following perturbation series for the surface displacement and velocity field:

$$\left.\begin{aligned} \eta &= \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \dots, \\ \boldsymbol{u} &= \boldsymbol{u}_0 + \varepsilon\boldsymbol{u}_1 + \varepsilon^2\boldsymbol{u}_2 + \dots, \\ w &= w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots . \end{aligned}\right\} \tag{2.5}$$

Substituting these expressions into (2.3) and (2.4) and collecting the terms of like-order result in

$$\eta_{0t} + \frac{C_p^2}{g}(\nabla\cdot\boldsymbol{u}_0) + \varepsilon\left[\eta_{1t} + \frac{C_p^2}{g}(\nabla\cdot\boldsymbol{u}_1) + \nabla\left(\frac{C_p^2}{g}\right)\cdot\boldsymbol{u}_0 + \nabla\cdot(\eta_0\boldsymbol{u}_0)\right] = O\left(\varepsilon^2\right), \tag{2.6}$$

$$
\begin{aligned}
&C_pC_g\boldsymbol{u}_{0t} + gC_p^2\nabla\eta_0 - \frac{C_p(C_p-C_g)}{k^2}\nabla(\nabla\cdot\boldsymbol{u}_{0t}) \\
&\quad +\varepsilon\left\{C_pC_g\boldsymbol{u}_{1t} + gC_p^2\nabla\eta_1 - \frac{C_p(C_p-C_g)}{k^2}\nabla(\nabla\cdot\boldsymbol{u}_{1t})\right\} \\
&\quad +\varepsilon\left\{C_p^2\nabla\left[\eta_0 w_{0t} + \tfrac{1}{2}(\boldsymbol{u}_0\cdot\boldsymbol{u}_0 + w_0^2)\right] - \nabla\left[\frac{C_p(C_p-C_g)}{k^2}\right](\nabla\cdot\boldsymbol{u}_{0t})\right\} = O\left(\varepsilon^2\right).
\end{aligned}
\tag{2.7}
$$

For the zeroth-order equations seeking periodic solutions $\eta_0 = a_0 \cdot \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} \pm \omega t)]$ and $\boldsymbol{u}_0 = \boldsymbol{b}_0 \cdot \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} \pm \omega t)]$ (here $\boldsymbol{x} = (x, y)$ is the horizontal position vector, a_0 and $\boldsymbol{b}_0$ are arbitrary scalar and vectorial constants, and $i^2 = -1$) yields

$$
\omega^2 = k^2C_p^2, \quad \boldsymbol{u}_0 = \mp\frac{g}{\omega}\boldsymbol{k}\eta_0 = \frac{g}{k^2C_p^2}\nabla\eta_{0t}, \tag{2.8}
$$

where the bold face $\boldsymbol{k}$ is the wave-number vector. The $\mp$ sign in the second equation does not cause any ambiguity as that particular approximation is used to evaluate the nonlinear term $\boldsymbol{u}_0\cdot\boldsymbol{u}_0$ only, and the last expression is used in evaluating $\eta_0\boldsymbol{u}_0$. To a first approximation the vertical component of the velocity at the still water level w_0 is η_{0t}, as dictated by the kinematic boundary condition. Then, using equation (2.8), the linearized kinematic boundary condition and the linearized continuity equation it is possible to express all the nonlinear and linear shoaling terms appearing in (2.6) and (2.7) as follows.

$$
\left.
\begin{aligned}
\boldsymbol{u}_0\cdot\boldsymbol{u}_0 = \frac{g^2}{C_p^2}\eta_0^2, \quad w_0^2 = (\eta_{0t})^2 = (\pm\mathrm{i}\omega\eta_0)^2 = -k^2C_p^2\eta_0^2, \\
\eta w_{0t} = \eta\eta_{0tt} = -k^2C_p^2\eta_0^2, \quad \eta_0\boldsymbol{u}_0 = \frac{g}{k^2C_p^2}\eta_0\nabla\eta_{0t}, \quad \nabla\cdot\boldsymbol{u}_{0t} = -\frac{g}{C_p^2}\eta_{0tt} = gk^2\eta_0.
\end{aligned}
\right\}
\tag{2.9}
$$

At this stage it is possible to facilitate perturbation solutions by treating the above zeroth-order expressions as known and using them in (2.6) and (2.7). However, our aim is not to construct weak perturbation solutions but to derive a nonlinear wave equation that can describe the wave evolutions under various incident wave conditions. To do this it is necessary to diverge from the usual perturbation analysis and go back to equation (2.5) for expressing the zeroth- and first-order quantities in terms of the original variables:

$$
\left.
\begin{aligned}
\eta_0 + \varepsilon\eta_1 = \eta + O\left(\varepsilon^2\right), \quad \varepsilon\eta_0 = \varepsilon\eta + O\left(\varepsilon^2\right), \\
\boldsymbol{u}_0 + \varepsilon\boldsymbol{u}_1 = \boldsymbol{u} + O\left(\varepsilon^2\right), \quad \varepsilon\boldsymbol{u}_0 = \varepsilon\boldsymbol{u} + O\left(\varepsilon^2\right), \\
w_0 + \varepsilon w_1 = w + O\left(\varepsilon^2\right), \quad \varepsilon w_0 = \varepsilon w + O\left(\varepsilon^2\right),
\end{aligned}
\right\}
\tag{2.10}
$$

Using (2.9) in (2.6) and (2.7) for the terms proportional to ε and employing the expressions in (2.10) yields

$$
\eta_t + \frac{C_p^2}{g}(\nabla\cdot\boldsymbol{u}) + \varepsilon\frac{\nabla\left(C_p^2\right)}{k^2C_p^2}\cdot\nabla\eta_t + \varepsilon\frac{g}{k^2C_p^2}\nabla\cdot(\eta\nabla\eta_t) = O\left(\varepsilon^2\right), \tag{2.11}
$$

$$
\begin{aligned}
&C_pC_g\boldsymbol{u}_t + gC_p^2\nabla\eta - \frac{C_p(C_p-C_g)}{k^2}\nabla(\nabla\cdot\boldsymbol{u}_t) \\
&\quad +\tfrac{1}{2}\varepsilon g^2\left(1 - 3\frac{k^2C_p^4}{g^2}\right)\nabla\left(\eta^2\right) - \varepsilon g\frac{\nabla\left[C_p^3(C_p-C_g)\right]}{C_p^2}\eta = O\left(\varepsilon^2\right).
\end{aligned}
\tag{2.12}
$$

Note that the perturbation technique serves simply as a device for establishing approximate replacements for the terms $O(\varepsilon)$. Some comments on the nonlinear term of (2.12) are warranted. The term is made up of three different contributions; namely, the terms proportional to $\boldsymbol{u}\cdot\boldsymbol{u}$, w^2 and ηw_t. The first of these is directly proportional to η^2, the last two terms are important only in deep water and therefore both have the coefficient $k^2C_p^4/g^2$, which tends to unity in deep water ($C_p^4 = g^2/k^2$) but negligibly small in shallow water ($C_p^4 = g^2h^2$).

The next step is to eliminate the inertia term from the momentum equation by performing cross-differentiations as it is not permissible to use the approximate zeroth-order relations for replacing the zeroth-order terms themselves. The algebra involved is lengthy but straightforward and therefore it is sufficient to give an outline only. Multiply (2.12) by C_p/gC_g and take the divergence of the resulting equation, noting that C_p, C_g and k are all spatially varying quantities. Differentiate the continuity equation with respect to time and use it to replace the inertia term appearing in the equation obtained in the previous step. The result is

$$-\eta_{tt} + \frac{C_p^3}{C_g}\nabla^2\eta + \varepsilon\left[\frac{C_p}{C_g}\nabla(C_pC_g) - 3\frac{(C_p - C_g)}{C_g}\nabla\left(C_p^2\right)\right]\nabla\eta$$
$$+\varepsilon g\left[1 + \frac{1}{2}\frac{C_p}{C_g}\left(1 - 3\frac{k^2C_p^4}{g^2}\right)\right]\nabla^2\left(\eta^2\right) - \frac{C_p^2(C_p - C_g)}{gk^2C_g}\nabla^2(\nabla\cdot\boldsymbol{u}_t) = O\left(\varepsilon^2\right), \tag{2.13}$$

where the zeroth-order relations have been invoked in re-expressing the linear shoaling terms.

The final step is to express the last term in (2.13) in terms of the surface displacement. Differentiating (2.11) with respect to time and then applying the gradient operator twice give the desired expression that can be used in (2.13) to obtain

$$C_g\eta_{tt} - C_p^3\nabla^2\eta - \frac{(C_p - C_g)}{k^2}\nabla^2\eta_{tt} - C_p\nabla(C_pC_g)\cdot\nabla\eta$$
$$-\tfrac{3}{2}gC_p\left(3 - 2\frac{C_g}{C_p} - \frac{k^2C_p^4}{g^2}\right)\nabla^2\left(\eta^2\right) = 0, \tag{2.14}$$

in which the bookkeeping parameter ε has been removed from the nonlinear and linear shoaling terms and $O(\varepsilon^2)$ is set to zero. Equation (2.14) can describe the combined effects of nonlinear refraction and diffraction on a specified incident wave field as it propagates over gently varying depth. A three-time-level centred finite-difference scheme can be adopted for the numerical solution of the one-dimensional form of (2.14). For the directional case, it is again possible to use a similar scheme by introducing an iterative approach. Demonstrative numerical simulations will be reported separately.

3. Applicable band-width of wave equations

In linearized form (2.14) is not identical with the time-dependent form of the mild-slope equation of Smith & Sprinks (1975); however, it can be manipulated into the conventional form of the mild-slope equation. First, drop out the nonlinear term and divide the equation by C_p. Replace the time derivatives of the dispersion term (the third term) by $-\omega^2 = -k^2C_p^2$ so that it becomes $C_p(C_p - C_g)\nabla^2\eta$ and can be combined with the second term to produce $-(C_pC_g)\nabla^2\eta$. Finally, re-write $C_g\eta_{tt}/C_p$ as $\eta_{tt} - (1 - C_g/C_p)\eta_{tt}$, which, after replacing the second η_{tt} with $-\omega^2\eta$, becomes $\eta_{tt} + \omega^2(C_p - C_g)\eta/C_p$. The resulting equation is then

$$\eta_{tt} + \omega^2 \left(\frac{C_p - C_g}{C_p}\right)\eta - \nabla(C_pC_g\nabla\eta) = 0, \tag{3.1}$$

which is the mild-slope equation in its time-dependent form as proposed by Smith & Sprinks (1975). In view of this correspondence it should be appropriate to name equation (2.14) as the *time-dependent nonlinear mild-slope equation*.

Since the linearized form of (2.14) is *not* the same as Smith & Sprinks's equation it is expected that the dispersion characteristics of the two equations should differ. Considering for simplicity only the one-dimensional case and an incident wave with an arbitrary wave-number k_a and a corresponding phase celerity C_a, it is quite straightforward to obtain the following relations for the linearized forms of (2.14) and (3.1) respectively:

$$C_a^2 = C_p^3 \bigg/ \left[C_g + \frac{k_a^2}{k^2}(C_p - C_g)\right], \tag{3.2}$$

$$C_a^2 = C_pC_g + \frac{k^2}{k_a^2}C_p(C_p - C_g), \tag{3.3}$$

where C_p, C_g and k are the phase and group velocities and the wave-number prescribed *a priori* in equations (2.14) and (3.1) for a fixed dominant frequency ω while k_a is the arbitrary wave-number which is free to take on any value ranging from *zero* to *infinity*. Figure 1*a* shows the relative phase velocity errors for equations (3.2) and (3.3) for a prescribed k which corresponds to shallow water waves ($kh = 1$). The exact phase speed $(C_a)_e$ for the arbitrary wave-number k_a is computed from the linear theory dispersion relation $(C_a)_e = (g/k_a \tanh k_ah)^{1/2}$. A similar comparison is made in figure 1*b* for a prescribed k which corresponds to deep water waves ($kh = 5$). When k_a is equal to the prescribed k in (3.2) and (3.3) the agreement is perfect but as the difference between k_a and k gets larger the approximation becomes poorer. Regardless of the selected k the conventional mild-slope equation exhibits very undesirable properties as k_a tends to zero. For shallow water waves, as is seen in figure 1*a*, equation (2.14) has very broad applicable spectral width, ranging from zero to the wave-numbers as much as three times the prescribed wave number k. Although for deep water waves the equation is good in a somewhat narrower range it is still better than the conventional mild-slope equation, as figure 1*b* reveals.

4. Degenerate forms of time-dependent nonlinear mild-slope equation

We shall now consider the special forms of (2.14) when the water depth is relatively shallow and deep. Let us begin with the case of very shallow water so that $C_g \simeq C_p \simeq (gh)^{1/2}$. Introducing this approximation to (2.14) results in

$$\eta_{tt} = g\nabla\cdot(h\nabla\eta) + \tfrac{3}{2}g\nabla^2(\eta^2), \tag{4.1}$$

which may be shown to be the combined form of Airy's non-dispersive nonlinear wave equations for varying depth, correct to the second-order in nonlinearity. Note that in evaluating the coefficient of the nonlinear terms we have neglected the contribution of $k^2C_p^4/g^2$ as it is quite small for shallow depths. Equation (4.1) does not seem to

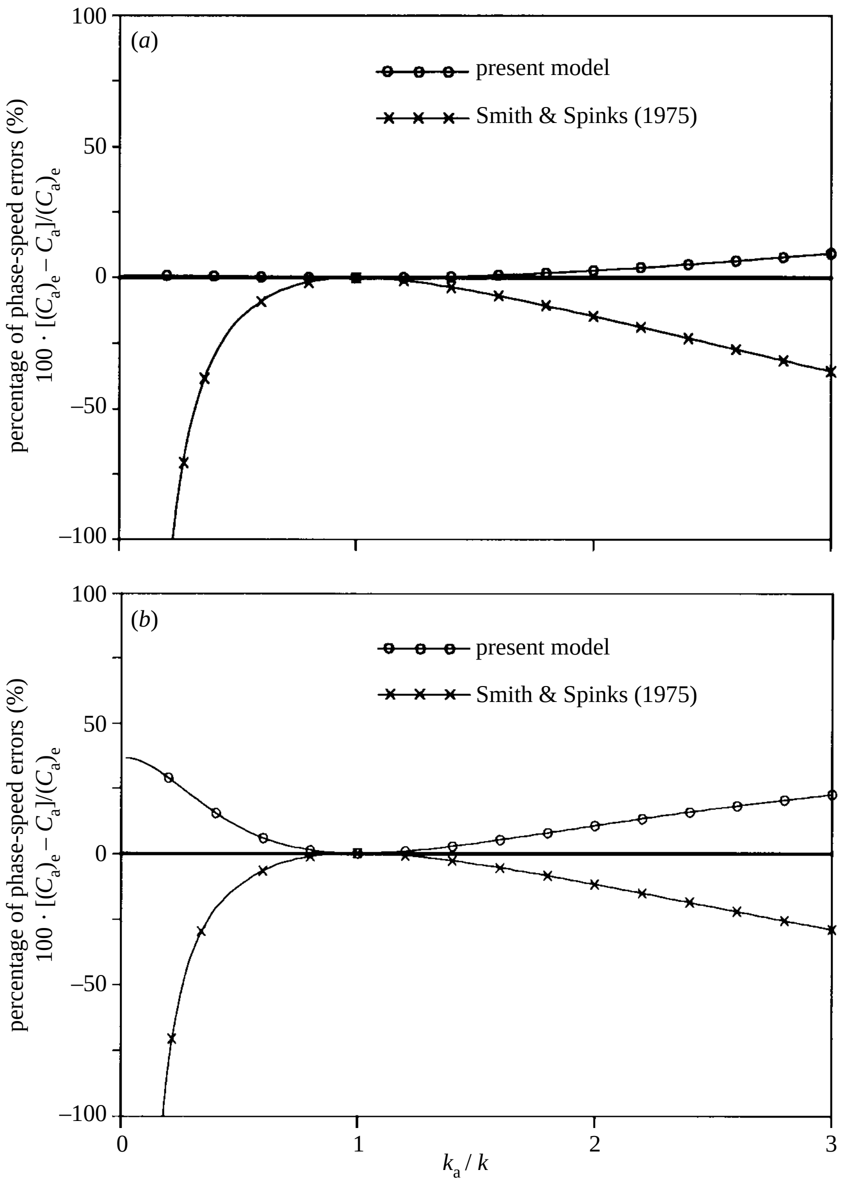

Figure 1. (a) Percentage of the phase speed errors for the present wave equation (∘) and the time-dependent mild-slope equation (×) of Smith & Sprinks (1975) in the vicinity of a prescribed wave number k which corresponds to relatively long waves ($kh = 1$). Note when $k_a = k$ the agreement is perfect for the both equations. (b) Same as in figure 1a but for a prescribed wave-number k which corresponds to deep water waves ($kh = 5$).

have been stated before but it is a trivial exercise to obtain it from Airy's equations by following the procedure used in constructing (2.14) from (2.1) and (2.2).

Retaining the lowest-order dispersion in C_p and C_g should result in the Boussinesq equation. Although not commonly used, the combined (singular) form of the Boussinesq equations for a constant depth has been given in the original work as reviewed by Miles (1980). Taking the first two terms of the Taylor series expansion of the exact linear theory dispersion relation

$$C_p \simeq (gh)^{1/2}\left(1 - \tfrac{1}{6}k^2h^2\right), \quad C_g \simeq (gh)^{1/2}\left(1 - \tfrac{1}{2}k^2h^2\right), \tag{4.2}$$

substituting these approximate expressions in (2.14) and invoking the zeroth-order relation $\eta_{tt} = -\omega^2\eta = -k^2C_p^2\eta = C_p^2\nabla^2\eta$ in the dispersion term, one obtains

$$\eta_{tt} = gh\nabla^2\eta + \tfrac{1}{3}gh^3\nabla^2\left(\nabla^2\eta\right) + \tfrac{3}{2}g\nabla^2\left(\eta^2\right), \tag{4.3}$$

in which the higher-order dispersion terms $O(k^4h^4)$ have been neglected in accordance with the classical Boussinesq approximations. In its one-dimensional form equation (4.3) is the same as the combined Boussinesq equation for constant depth (Miles 1980).

For deep water waves $C_p \simeq (g/k)^{1/2}$ and the group velocity is half the phase velocity $C_g \simeq C_p/2$. Equation (2.14) in this case reads

$$\eta_{tt} = 2\frac{g}{k}\nabla^2\eta + \frac{1}{k^2}\nabla^2\eta_{tt} + 3g\nabla^2\left(\eta^2\right), \tag{4.4}$$

where the linear shoaling term has been dropped out. In the literature it is not possible to find a corresponding equation; however, any second-order nonlinear wave equation for deep water waves must be able to produce the second-order Stokes waves. This can be checked fairly easily by seeking a solution of the form $\eta = a \cdot \cos[(\boldsymbol{k}\cdot\boldsymbol{x} \pm \omega t)] + b \cdot \cos[2(\boldsymbol{k}\cdot\boldsymbol{x} \pm \omega t)]$. Substituting this expression in (4.4) and retaining the nonlinear terms up to $O(\varepsilon^2)$ give, respectively, for the zeroth- and first-order terms

$$\omega^2 = gk, \quad b = \tfrac{1}{2}ka^2, \tag{4.5}$$

proving that (4.4) admits the second-order permanent Stokes waves as analytical solutions. Note the waves need not propagate in a direction parallel to a horizontal axis; the solution is valid for arbitrary propagation directions. While the second-order Stokes waves possess no amplitude dispersion the evolution equation (4.4) and its general form (2.14) always give rise to amplitude dispersion because of the η^2 term which generates interacting higher harmonic components from the initial wave form $\eta = a \cdot \cos[(\boldsymbol{k}\cdot\boldsymbol{x} \pm \omega t)] + b \cdot \cos[2(\boldsymbol{k}\cdot\boldsymbol{x} \pm \omega t)]$. For this reason, the solutions of such evolution equations produce partially correct higher-order amplitude dispersion effects. Considering this point, in § 7, the numerical solutions are compared with the third-order phase celerity of the Stokes theory.

5. A one-way propagation wave model

While equation (2.14) in its present form is quite suitable for any numerical modelling of certain wave phenomenon of interest there are possibilities of elaborating it further, such as considering the unidirectional propagation in the positive x-direction only. The reason for taking up the analysis of such a simplified case lies in the attractive form of the KdV equation, which will be recovered as a special case. Let us begin with the one-dimensional form of (2.14):

$$C_g\eta_{tt} - C_p^3\eta_{xx} - \frac{(C_p - C_g)}{k^2}\eta_{xxtt} - C_p(C_pC_g)_x\eta_x - \tfrac{3}{2}gC_p\left(3 - 2\frac{C_g}{C_p} - \frac{k^2C_p^4}{g^2}\right)\left(\eta^2\right)_{xx} = 0. \tag{5.1}$$

Introduce a new coordinate system (σ, τ) moving in the positive x-direction with the phase speed C_p so that evolutions of the wave form in time is slow, permitting us to write (Mei 1983, p. 549)

$$\sigma = x - C_p t, \quad \tau = \varepsilon t, \tag{5.2}$$

where ε is a small parameter signifying the slow changes of wave form in this moving coordinate system. Expressing (5.1) in the new coordinate system, rearranging, and dropping the terms $O(\varepsilon^2)$ as being higher-order than we intend to retain result in

$$\begin{aligned}-2\varepsilon C_p C_g \eta_{\sigma\tau} - C_p^2(C_p - C_g)\eta_{\sigma\sigma} - \frac{C_p^2(C_p - C_g)}{k^2}\eta_{\sigma\sigma\sigma\sigma} \\ +2\varepsilon\frac{C_p(C_p - C_g)}{k^2}\eta_{\sigma\sigma\sigma\tau} + \varepsilon\left[C_p C_g (C_p)_\sigma - C_p(C_p C_g)_\sigma\right]\eta_\sigma \\ -\varepsilon\frac{C_p(C_p - C_g)}{k^2}(C_p)_\sigma \eta_{\sigma\sigma\sigma} - \tfrac{3}{2}\varepsilon g C_p\left(3 - 2\frac{C_g}{C_p} - \frac{k^2 C_p^4}{g^2}\right)\left(\eta^2\right)_{\sigma\sigma} = 0,\end{aligned} \tag{5.3}$$

where the terms proportional to the spatial derivative of the phase and group speeds are also labelled by ε to emphasize that they are an order smaller compared with the zeroth-order relations.

The form of (5.3) suggests an integration with respect to σ; however it cannot be done by simply removing a subscript from each term because C_p, C_g and k are all spatially varying quantities and therefore they must be treated with due care in such an integration. The terms proportional to ε need not be considered since any variation of C_p, C_g and k in these terms can be neglected as being higher-order. The zeroth-order terms on the other hand require the care we indicated. Observing this precaution, the integration of (5.3) with respect to σ gives

$$\begin{aligned}-2\varepsilon C_p C_g \eta_\tau - C_p^2(C_p - C_g)\eta_\sigma + 2\varepsilon\frac{C_p(C_p - C_g)}{k^2}\eta_{\sigma\sigma\tau} \\ -\frac{C_p^2(C_p - C_g)}{k^2}\eta_{\sigma\sigma\sigma} + \varepsilon\frac{\left[3C_p(C_p - C_g)(C_p)_\sigma + C_p^2(C_p - C_g)_\sigma\right]}{k^2}\eta_{\sigma\sigma} \\ +\varepsilon C_p\left[\tfrac{3}{2}C_p^2 - 2(C_p C_g)\right]_\sigma \eta - \tfrac{3}{2}\varepsilon g C_p\left(3 - 2\frac{C_g}{C_p} - \frac{k^2 C_p^4}{g^2}\right)\left(\eta^2\right)_\sigma = 0.\end{aligned} \tag{5.4}$$

The coordinate transformation has served its purpose of extracting a wave equation for the right-moving waves from the one-dimensional equation (5.1) and the inverse transformation may now be carried out to go back to the usual fixed coordinate system. The inverse transformations for η_σ, $\varepsilon\eta_\tau$, etc., may be constructed easily with the help of (5.2) and are skipped here. In the original fixed coordinate system equation (5.4) becomes

$$\begin{aligned}C_g\eta_t + \tfrac{1}{2}C_p(C_p + C_g)\eta_x - \frac{(C_p - C_g)}{k^2}\eta_{xxt} - \frac{C_p(C_p - C_g)}{2k^2}\eta_{xxx} \\ +\tfrac{1}{2}\left[C_p(C_g)_x + (C_p - C_g)(C_p)_x\right]\eta + \tfrac{3}{4}g\left(3 - 2\frac{C_g}{C_p} - \frac{k^2 C_p^4}{g^2}\right)\left(\eta^2\right)_x = 0,\end{aligned} \tag{5.5}$$

in which the linear shoaling terms are collected together by using the zeroth-order relation $\eta_{xx} = -k^2\eta$.

Equation (5.5) describes the weakly nonlinear wave evolutions of a narrow-banded unidirectional wave field centred around the dominant wave frequency $\omega = kC_p$.

There is no restriction on the depth these waves must travel, it may range from nearly zero to infinity. The equation also models the linear shoaling in exact form (for monochromatic waves with $\omega = kC_p$) so long as the second and higher derivatives and squares of derivatives of the phase and group velocities (or the bottom slope) are negligible. All the degenerate forms of (2.14), when reduced to the unidirectional forms, are equally applicable to (5.5) as well. Let us briefly examine them for the sake of completeness.

6. Degenerate forms of one-way propagation model

For very shallow water waves the phase and group velocity are nearly the same and $C_p \simeq C_g \simeq (gh)^{1/2}$. When this approximation is used in (5.5) it becomes a nonlinear, non-dispersive wave equation, which may be recognized as the combined unidirectional form of Airy's nonlinear shallow water equations:

$$\eta_t + C_0 \left[\eta_x + \frac{h_x}{4h}\eta + \frac{3}{4h}\left(\eta^2\right)_x \right] = 0, \tag{6.1}$$

where $C_0 = (gh)^{1/2}$. Inclusion of a weak-dispersion effect in C_p and C_g as given in (4.2) produces the KdV equation for a gently varying depth:

$$\eta_t + C_0 \left[\eta_x + \frac{h_x}{4h}\eta + \frac{h^2}{6}\eta_{xxx} + \frac{3}{4h}\left(\eta^2\right)_x \right] = 0. \tag{6.2}$$

The dispersion term need not be in the form given above; various alternatives are possible by using the zeroth-order relations as enumerated in Mei (1983). Indeed (5.5) produces a KdV equation containing both η_{xxt} and η_{xxx} as dispersion terms. The equation has been manipulated into the more familiar form given in (6.2).

If the depth, relative to the wavelength, is large $C_p \simeq (g/k)^{1/2}$ and $C_g \simeq C_p/2$. For this special case equation (5.5) becomes

$$\eta_t + C_p \left[\frac{3}{2}\eta_x - \frac{1}{k^2 C_p}\eta_{xxt} - \frac{1}{2k^2}\eta_{xxx} + \frac{3}{2}k\left(\eta^2\right)_x \right] = 0, \tag{6.3}$$

which may be shown to admit the unidirectional second-order Stokes waves as analytical solutions.

7. Numerical simulations

The form of equation (5.5) permits the use of standard finite-difference techniques of solving the KdV-type equations. A three-time-level implicit scheme appears to be the best choice as it reduces the handling of the nonlinear term to a quasi-linear form and allows relatively large spatial and time steps. The resulting discretized equations constitute a tridiagonal matrix, requiring the incoming and outgoing boundary conditions be specified in the first and last rows, respectively. Care should be observed in discretizing η_{xxx} at the node next to the incoming boundary, otherwise accurate propagation of waves is hindered considerably. In the present scheme this was accomplished by simply invoking the zeroth-order relation $\eta_{xx} = -k^2\eta$ so that $\eta_{xxx} = -k^2\eta_x$, which proved to be a very good approximation even for incident waves of limiting height. The initial condition used in all the computations presented here was the state of rest, which was imposed by setting the first two time level values

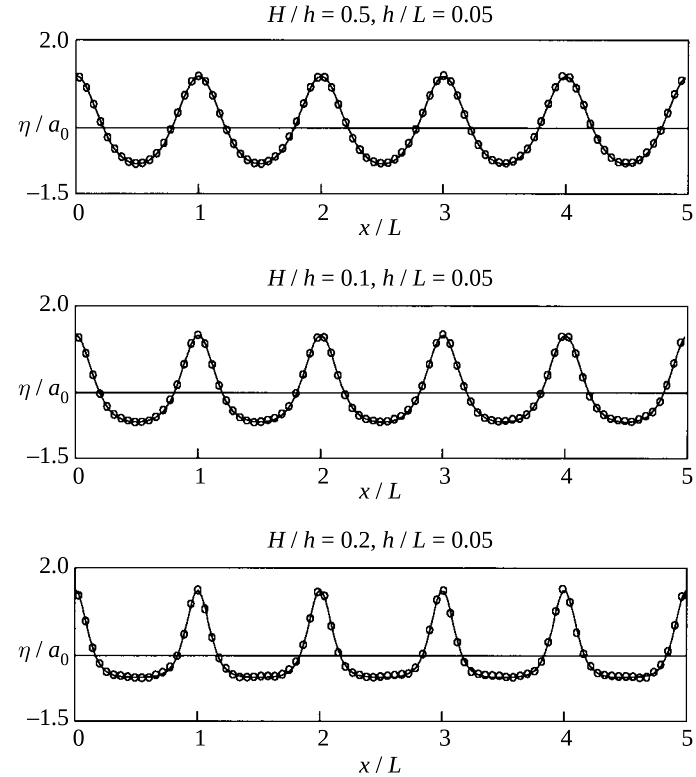

Figure 2. Cnoidal wave simulations. Wave parameters are selected from experimentally reproduced cnoidal wave forms (Goring & Raichlen 1980): analytical expression (-) and numerical solution of the unidirectional wave model (∘).

to zero throughout the computational domain. It is of course possible to commence a computation from a different initial configuration by setting the first two time level values in accordance with the desired initial condition. The prescribed time-dependent boundary data is introduced into the computational domain by setting the new time level value of the first node to the specified surface displacement at each time step. Overall, the resulting scheme has been found to be quite robust and accurate.

The first case is the simulation of cnoidal waves. Since physically realizable permanent cnoidal waves exist only for definite H/h, h/L and m values (H: wave height, h: water depth, L: wavelength and m: elliptic parameter) three sample wave forms from Goring & Raichlen's (1980) experiments were selected: $H/h = 0.05$, $m = 1 - 2.15 \times 10^{-1}$ (CN2); $H/h = 0.1$, $m = 1 - 5.21 \times 10^{-2}$ (CN3); $H/h = 0.2$, $m = 1 - 5.38 \times 10^{-3}$ (CN4) with $h/L = 0.05$ for all the cases. The time and spatial resolutions were $\Delta t = T/35$, $\Delta x = L/35$; the computations were done for 15 wave periods, allowing the wave field to develop fully in the entire computational domain. As is seen from figure 2 the model simulations show excellent agreement with the theory and clearly prove the ability of the proposed wave equation to propagate permanent cnoidal waves.

The Stokes waves are the deep water counterpart of cnoidal waves and may be simulated equally accurately by the present wave model. Figure 3 depicts three cases for Stokes-type waves with different kH and h/L values which vary from relatively

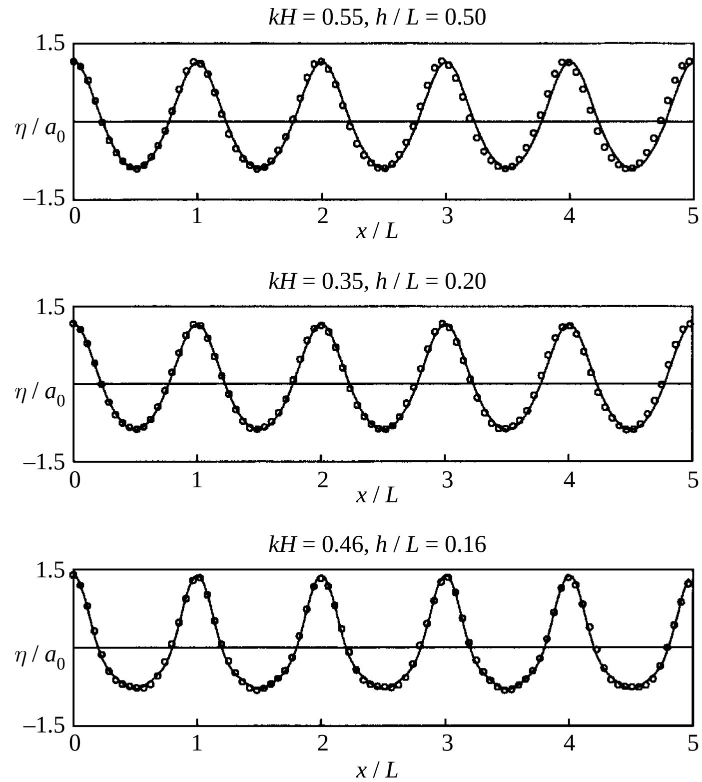

Figure 3. Second-order Stokes wave simulations. Wave parameters are selected from experimentally reproduced Stokes-type waves (Morrison & Crooke 1953): Stokes's second-order solution (-) and numerical solution of the unidirectional wave model (○).

deep to shallow depths. These cases are selected because they were reproduced in the laboratory experiments (Morrison & Crooke 1953) and found to agree well with the analytical solutions. Note the last case ($h/L = 0.16$) corresponds virtually to shallow water waves; it delineates the applicable limit of the Stokes theory according to Morrison & Crooke's experiments. In the numerical simulations the spatial and time resolutions and the simulation duration were taken to be the same as the previous case. The agreement of the numerical solution of (5.5) with the theoretical solution and consequently with the experimental measurement for each case is remarkably good, regardless of the $kH - h/L$ value. Small phase discrepancies are due to the use of the third-order Stokes wave celerity in the analytical solutions. This point has been discussed in § 4 in some detail.

The linear shoaling characteristics of (5.5) are checked for uniformly varying depth with a constant slope of 1:50. The water depth in the deep section is 30 m and after a distance of 1000 m it reduces to 10 m. Incident waves of sinusoidal form for three different periods, which correspond respectively to deep, intermediate and shallow water cases, are imposed at the incoming boundary. The initial wave amplitude is taken as 1 m for simplicity. In figure 4, the variation of wave amplitude for each case, computed according to the constancy of energy flux (linear theory), is compared with the numerical results obtained from the linearized version of (5.5). The agreement is almost perfect; this is to be expected because it can analytically be shown (see the procedure devised by Madsen & Sørensen, 1992) that the linearized version of (5.5),

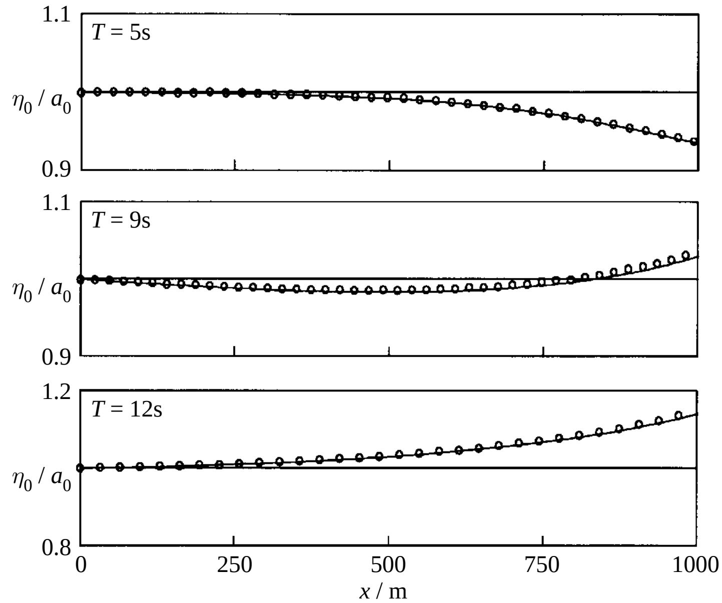

Figure 4. Demonstration of the linear shoaling characteristics of equation (5.5) for varying depth with a constant bed slope of 1:50. Incident waves are sinusoidal with initial amplitude 1 m at 30 m water depth. For each case, variation of the wave amplitude with distance is computed according to the constancy of energy flux (-) and compared with numerical solution of the unidirectional wave model (○).

like (2.14), incorporates exact linear shoaling for gentle slopes when the incident wave frequency coincides with the prescribed frequency of the model. Additional tests with steeper slopes, such as 1:25, and with curved bathymetry have further confirmed the accuracy of the model; however these comparisons had to be excluded due to space limitations.

Finally, it is worthwhile to emphasize that the applicability of the model equations is not restricted to periodic waves; narrow-banded random waves may also be simulated accurately. Such computations will be reported separately.

8. Concluding remarks

A nonlinear-dispersive wave equation, named as the *time-dependent nonlinear mild-slope equation*, has been derived. The degenerate forms of this equation yield all the well-known linear and nonlinear wave equations and in deep water the equation admits the second-order Stokes waves as analytical solutions. The unidirectional form of the model equation also retains all the nonlinear characteristics of its generic and likewise embodies the existing linear and nonlinear wave equations in its class as special cases. A further extension of this work would be the derivation of a KP-type equation for weakly nonlinear weakly directional waves over arbitrary depths by retaining a weak-directionality in the one-way wave propagation model.

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