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TECHNICAL NOTE

NOTE ON CONSERVATION EQUATIONS FOR NONLINEAR SURFACE WAVES

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Abstract—Euler's equations of motion in conjunction with the dynamic boundary condition are manipulated to obtain exact (and approximate) alternative momentum equations for nonlinear irrotational surface waves. The Airy and Boussinesq equations are re-derived as demonstrative examples. A fully nonlinear version of the improved Boussinesq equations is presented as a new application of the proposed equations. Further use of the equations in developing depth-integrated wave models, which are not necessarily restricted to finite depths, is also pointed out.

1. INTRODUCTION

A wave model is in essence the result of a conversion from the three-dimensional governing equations to the two-dimensional wave equations. Such a process requires the specification of the vertical dependence of the velocity or potential field, which in turn is used in the governing equations for obtaining the corresponding wave equations. The implementation of this stratagem presents no difficulties for linear models; however, it always gives rise to cumbersome expressions for nonlinear equations. Here, an appealing alternative, which bypasses the burdens of the direct use of the governing equations, is presented in the form of a momentum equation with quite manageable nonlinear terms. The equation may be used conveniently for the development of highly nonlinear new wave models, as demonstrated for a sample case.

The outline of the paper is as follows. The governing equations and boundary conditions appropriate to the problem in hand are recapitulated in the following section. The depth-integrated continuity equation is re-derived in Section 3 for the sake of completeness. In Section 4 Euler's equations for irrotational flow are stated first, and then an exact alternative momentum equation is derived. With further manipulations a variant of this equation is also obtained. The approximate versions of these equations are given in Section 5. Section 6 offers remarkably short derivations of the Airy and Boussinesq equations as simple demonstrations and gives a fully nonlinear version of the improved Boussinesq equations expressed in terms of the velocity at the still water. In closing, possibilities of developing new wave models using the conservation equations derived are pointed out.

2. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

The governing equations of an inviscid, irrotational flow are given by the continuity equation, Euler's equations, and irrotationality conditions:

$$\nabla\cdot\mathbf{u} + \frac{\partial w}{\partial z} = 0 \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + w\frac{\partial \mathbf{u}}{\partial z} = -\frac{1}{\rho}\nabla p \tag{2}$$

$$\frac{\partial w}{\partial t} + (\mathbf{u}\cdot\nabla)w + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{3}$$

$$\frac{\partial \mathbf{u}}{\partial z} = \nabla w, \quad \frac{\partial \mathbf{u}}{\partial y} = \frac{\partial w}{\partial x} \tag{4}$$

where $\mathbf{u}$, w are respectively the horizontal velocity vector and vertical velocity component, p is the pressure, and g is the gravitational acceleration. Bold symbols indicate vectors with x- and y-components only, that is $\mathbf{u} = (u,v)$ and $\mathbf{x} = (x,y)$. The two-dimensional gradient operator, $(\partial/\partial x, \partial/\partial y)$, is denoted by ∇.

The boundary conditions for a free surface flow bounded by an impermeable rigid bottom of arbitrary shape may be stated as follows:

$$p = p_s \qquad \text{at } z = \eta(\mathbf{x},t) \tag{5}$$

$$w = \frac{\partial \eta}{\partial t} + \mathbf{u}\cdot\nabla\eta \qquad \text{at } z = \eta(\mathbf{x},t) \tag{6}$$

$$\mathbf{u}\cdot\nabla h + w = 0 \qquad \text{at } z = -h(\mathbf{x}) \tag{7}$$

in which p_s is a specified surface pressure, $\eta(\mathbf{x},t)$ is the free surface elevation, $h(\mathbf{x})$ is the local water depth as measured from the still water level. The origin of the coordinate system is taken at the still water level with positive z-axis pointing upward. The first condition states that the pressure is known at the free surface (usually taken constant or zero). The second condition is the kinematic free surface condition which asserts that the particles on the surface remain there. Finally, Equation (7) is the bottom condition, expressing that the velocity normal to the bottom must vanish.

3. DEPTH-INTEGRATED CONTINUITY EQUATION

Integrating Equation (1) over the entire water depth yields

$$\int_{-h}^{\eta} (\nabla\cdot\mathbf{u})\,\mathrm{d}z + w(\eta) - w(-h) = 0 \tag{8}$$

Using the boundary conditions (6) and (7) for $w(\eta)$ and $w(-h)$, respectively, and invoking the Leibnitz rule, one obtains the well-known result

$$\frac{\partial \eta}{\partial t} + \nabla\cdot\left(\int_{-h}^{\eta} \mathbf{u}\,\mathrm{d}z\right) = 0 \tag{9}$$

which is exact. A weakly nonlinear version of Equation (9) follows from the Taylor series expansion of $\mathbf{u}$ at $z = 0$:

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left(\int_{-h}^{\eta} \mathbf{u} \, \mathrm{d}z + \eta \mathbf{u}_0 \right) = 0 \tag{10}$$

in which $\mathbf{u}_0$ is the horizontal velocity vector at the still water level $z = 0$.

4. AN ALTERNATIVE MOMENTUM EQUATION AND ITS VARIANT

Using the irrotationality conditions stated in Equation (4) it is a straightforward matter to cast the Euler equations into the following forms:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla (\mathbf{u} \cdot \mathbf{u} + w^2) = -\frac{1}{\rho} \nabla p \tag{11}$$

$$\frac{\partial w}{\partial t} + \frac{1}{2} \frac{\partial}{\partial z} (\mathbf{u} \cdot \mathbf{u} + w^2) = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \tag{12}$$

We proceed by integrating the vertical momentum equation from an arbitrary depth z to the free surface η:

$$\int_{z}^{\eta} \frac{\partial w}{\partial t} \, \mathrm{d}z + \frac{1}{2} (\mathbf{u} \cdot \mathbf{u} + w^2) \Big|_{z}^{\eta} = -\left(\frac{1}{\rho} p + gz \right) \Big|_{z}^{\eta} \tag{13}$$

The boundary condition stated in Equation (5) requires $p(\eta) = p_{\mathrm{s}}$ so that, from Equation (13), the pressure at an arbitrary depth z is

$$\frac{1}{\rho} p = \frac{1}{\rho} p_{\mathrm{s}} + g(\eta - z) + \int_{z}^{\eta} \frac{\partial w}{\partial t} \, \mathrm{d}z + \frac{1}{2} [(\mathbf{u}_{\mathrm{s}} \cdot \mathbf{u}_{\mathrm{s}} + w_{\mathrm{s}}^2) - (\mathbf{u} \cdot \mathbf{u} + w^2)] \tag{14}$$

where the variables evaluated at the free surface $z = \eta$ are denoted as $\mathbf{u}_{\mathrm{s}}$, w_{s}, while the variables at an arbitrary depth are left as before.

Substitute Equation (14) into Equation (11) to obtain the following momentum equation which, together with Equation (9), may be used in constructing wave models:

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left[g\eta + \int_{z}^{\eta} \frac{\partial w}{\partial t} \, \mathrm{d}z + \frac{1}{2} (\mathbf{u}_{\mathrm{s}} \cdot \mathbf{u}_{\mathrm{s}} + w_{\mathrm{s}}^2) + \frac{1}{\rho} p_{\mathrm{s}} \right] = 0 \tag{15}$$

The above equation is exact for an irrotational inviscid free surface flow and may be considered as the dynamic counterpart of the kinematic equation (9). At this stage some remarks are appropriate: the first term represents the fluid inertia in the horizontal directions at an arbitrary depth z, the second is the effect of hydrostatic pressure gradient due to the spatial variation of the free surface, the third is the so-called dispersion term arising from the vertical acceleration of the fluid column which may be considered as the non-hydrostatic contribution to the pressure. The terms in parentheses are the nonlinear contributions associated with the particle kinetic energies, and finally the last term is the effect of the specified surface pressure distribution.

Equation (15) may be further elaborated and put into a form which contains only surface variables. First we note that with the help of the Leibnitz rule the integral appearing in Equation (15) may be rewritten as

$$\nabla\left(\int_z^\eta \frac{\partial w}{\partial t}\right) = \int_z^\eta \frac{\partial}{\partial t}(\nabla w)\,\mathrm{d}z + \left(\frac{\partial w}{\partial t}\right)_{\mathrm{s}} \nabla\eta \tag{16}$$

Using the irrotationality condition, $\nabla w = \partial \mathbf{u}/\partial z$, and performing the integration yield

$$\nabla\left(\int_z^\eta \frac{\partial w}{\partial t}\,\mathrm{d}z\right) = \left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathrm{s}} - \frac{\partial \mathbf{u}}{\partial t} + \left(\frac{\partial w}{\partial t}\right)_{\mathrm{s}} \nabla\eta \tag{17}$$

which may be used in Equation (15) to obtain

$$\left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathrm{s}} + \left[g + \left(\frac{\partial w}{\partial t}\right)_{\mathrm{s}}\right]\nabla\eta + \frac{1}{2}\nabla(\mathbf{u}_{\mathrm{s}}\cdot\mathbf{u}_{\mathrm{s}} + w_{\mathrm{s}}^2) + \frac{1}{\rho}\nabla p_{\mathrm{s}} = 0 \tag{18}$$

where all the terms are expressed in terms of the surface quantities only. Note, in general, $(\partial\mathbf{u}/\partial t)_{\mathrm{s}} \neq \partial\mathbf{u}_{\mathrm{s}}/\partial t$. Equation (18), a variant of Equation (15), may also be used with Equation (9) in developing nonlinear wave models.

Equation (18) is of course directly recoverable from the Bernoulli equation evaluated at the free surface $z = \eta$. One must only be careful in taking the gradient of $(\partial\phi/\partial t)_{\mathrm{s}}$:

$$\nabla\left[\left(\frac{\partial\phi}{\partial t}\right)_{\mathrm{s}}\right] = \left[\frac{\partial(\nabla\phi)}{\partial t}\right]_{\mathrm{s}} + \left(\frac{\partial^2\phi}{\partial t\partial z}\right)_{\mathrm{s}}\nabla\eta = \left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathrm{s}} + \left(\frac{\partial w}{\partial t}\right)_{\mathrm{s}}\nabla\eta \tag{19}$$

in which ϕ is the velocity potential.

5. APPROXIMATE FORMS

Weakly nonlinear versions of the exact momentum equations (15) and (18) are now derived. The approximation is carried out by expressing these equations in terms of the variables at the still water level instead of the actual free surface. Introducing Taylor series expansions for the nonlinear terms in Equation (15) and keeping only the leading order terms gives

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left[g\eta + \int_z^0 \frac{\partial w}{\partial t}\,\mathrm{d}z + \eta\,\frac{\partial w_0}{\partial t} + \frac{1}{2}(\mathbf{u}_0\cdot\mathbf{u}_0 + w_0^2)\right] = 0 \tag{20}$$

where the subscript 0 marks the variables evaluated at the still water level. Note also that p_{s} is set to zero as is usual for problems without any surface disturbance.

Carrying out a similar procedure for Equation (18) gives

$$\frac{\partial \mathbf{u}_0}{\partial t} + \nabla\left[g\eta + \eta\,\frac{\partial w_0}{\partial t} + \frac{1}{2}(\mathbf{u}_0\cdot\mathbf{u}_0 + w_0^2)\right] = 0 \tag{21}$$

where use has been made of the approximation that in the vicinity of the still water level $\partial \mathbf{u}/\partial t \approx \partial \mathbf{u}_0/\partial t + z\partial^2 \mathbf{u}_0/\partial t \partial z$ so that $(\partial \mathbf{u}/\partial t)_s \approx \partial \mathbf{u}_0/\partial t + \eta \nabla(\partial w_0/\partial t)$. The replacement in the second term follows from the irrotationality condition $\partial \mathbf{u}_0/\partial z = \nabla w_0$.

6. APPLICATIONS

6.1. *Airy equations*

The Airy equations can readily be obtained from Equations (10) and (21) by simply setting $\mathbf{u} = \mathbf{u}_0$ in Equation (10) and $w_0 = 0$ in Equation (21):

$$\frac{\partial \eta}{\partial t} + \nabla \cdot [(h + \eta)\mathbf{u}_0] = 0 \tag{22}$$

$$\frac{\partial \mathbf{u}_0}{\partial t} + \nabla \left(g\eta + \frac{1}{2}\, \mathbf{u}_0 \cdot \mathbf{u}_0 \right) = 0 \tag{23}$$

The unconventional form of the advection term is due to the use of the irrotationality condition. Although $\mathbf{u}_0$ stands for the velocity vector at $z = 0$, it may be replaced by the conventional depth-averaged velocity since the vertical distribution of the horizontal velocity field is uniform in the Airy theory.

6.2. *Boussinesq equations*

The Boussinesq equations are weakly dispersive, weakly nonlinear equations, therefore it is sufficient to begin with any one of the approximate equations derived in the previous section. For demonstration purposes, instead of Equations (10) and (21) we shall begin with Equations (10) and (20). As these equations are valid for an arbitrary bathymetry we can derive the Boussinesq equations for varying depth. Due to the compact form of Equation (20) the entire procedure will be quite simple in comparison with the established techniques. Let the horizontal and vertical velocity components be expanded in power series in z:

$$\mathbf{u} = \sum_{n=0}^{\infty} z^n \mathbf{u}_{0n}(\mathbf{x}), \quad w = \sum_{n=0}^{\infty} z^n w_{0n}(\mathbf{x}) \tag{24}$$

where subscript 0 stands for the quantities evaluated at the still water level $z = 0$. In order that these expansions be admissible, they must satisfy the kinematic conditions, namely the bottom condition, continuity equation and irrotationality condition. Satisfying these conditions and keeping only the first two terms of the series expansion results in

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_0 - z\nabla[\nabla \cdot (h\mathbf{u}_0)] - \frac{1}{2} z^2 \nabla(\nabla \cdot \mathbf{u}_0) \\ w &= -\nabla \cdot (h\mathbf{u}_0) - z(\nabla \cdot \mathbf{u}_0) \end{aligned} \tag{25}$$

where $\mathbf{u}_0 = \mathbf{u}_{00}$ is the horizontal velocity vector at the still water level $z = 0$ and $h = h(\mathbf{x})$. Thus far, in the kinematic part of the development, we have followed the usual process. It is the dynamic part of the derivation that is shortened considerably by the use of the alternative momentum equation. Substituting Equation (25) into Equations (10) and (20) gives

$$\frac{\partial \eta}{\partial t} + \nabla\cdot\left\{(h+\eta)\mathbf{u}_0 + \frac{1}{2}h^2\nabla[\nabla\cdot(h\mathbf{u}_0)] - \frac{1}{6}h^3\nabla(\nabla\cdot\mathbf{u}_0)\right\} = 0 \tag{26}$$

$$\frac{\partial \mathbf{u}_0}{\partial t} + \nabla\left(g\eta + \frac{1}{2}\mathbf{u}_0\cdot\mathbf{u}_0\right) = 0 \tag{27}$$

which are the Boussinesq equations for varying bathymetry, expressed in terms of the velocity at the still water level (Peregrine, 1967). In evaluating Equation (20) the nonlinear terms proportional to the vertical velocity component have been neglected completely in accordance with the usual Boussinesq approximations. Note that the velocity variables at the arbitrary depth z cancelled each other out as a consequence of the satisfied irrotationality condition $\mathbf{u}_z = \nabla w$. Obviously, the nonlinear range of the Boussinesq equations may be improved considerably if one begins with Equations (9) and (15). Unlike standard approximations, the compact form of Equation (15) offers a numerically attractive alternative for high nonlinearity; such an application is considered next.

6.3. *Fully nonlinear improved Boussinesq equations*

Beji and Nadaoka (1996) introduced the concept of partial replacement for improving the dispersion characteristics of the Boussinesq equations. Accordingly, the second-order linear shoaling terms are partitioned by a simple algebraic manipulation and a part of these terms are re-expressed using a first-order relation. The result is a Boussinesq model with mixed dispersion terms that accommodates better dispersion characteristics. The same procedure may be applied to the Boussinesq model derived in the previous section. Furthermore, by taking advantage of the compact form of Equations (9) and (15) a fully nonlinear form of the improved version of Equations (26) and (27) may be derived. Substituting Equation (25) into Equations (9) and (15), applying the partial replacement procedure to the resulting continuity equation with a partition parameter β^* and finally using the first-order relation $\eta_t + \nabla\cdot(h\mathbf{u}_0) = 0$ for re-expressing only one part of the dispersion terms result in

$$\frac{\partial \eta}{\partial t} + \nabla\cdot\left[(h+\eta)\mathbf{u}_0 + \frac{\beta^*}{3}h\nabla\left(h\frac{\partial \eta}{\partial t}\right) - \Gamma\right] \tag{28}$$

$$+ \frac{(1+\beta^*)}{2}\nabla\cdot\left\{h^2\nabla[\nabla\cdot(h\mathbf{u}_0)] - \frac{1}{3}h^3\nabla(\nabla\cdot\mathbf{u}_0)\right\} = 0$$

$$\frac{\partial \mathbf{u}_0}{\partial t} + \nabla\left[g\eta + \frac{1}{2}(\mathbf{u}_s\cdot\mathbf{u}_s + w_s^2) - \Lambda\right] = 0 \tag{29}$$

where

$$\Gamma = \frac{1}{2}\eta^2\nabla[\nabla\cdot(h\mathbf{u}_0)] + \frac{1}{6}\eta^3\nabla(\nabla\cdot\mathbf{u}_0)$$

$$\Lambda = \eta\nabla\left(h\frac{\partial \mathbf{u}_0}{\partial t}\right) + \frac{1}{2}\eta^2\nabla\cdot\frac{\partial \mathbf{u}_0}{\partial t} \tag{30}$$

The surface velocities $\mathbf{u}_s$ and w_s are obtained from Equation (25) by setting $z = \eta$. Note if the higher-order nonlinear contributions Γ, Λ and w_s are completely neglected and $\mathbf{u}_s$ is approximated as $\mathbf{u}_0$ then setting the dispersion parameter $\beta^* = 0$ leads to the original equations derived in the previous section. On the other hand, keeping all these nonlinear contributions and choosing $\beta^* = -6/5$ give a fully nonlinear Boussinesq model with a linear dispersion relation that corresponds to the second-order Padé expansion of the linear theory dispersion relation. Furthermore, it may analytically be shown that the linear shoaling gradient obtained using the energy flux concept agrees perfectly with the linear shoaling gradient obtained from Equations (28) and (29), as in the model of Beji and Nadaoka (1996).

7. CONCLUDING REMARKS

An exact alternative momentum equation (15) and its variant (18) have been derived for the development of depth-integrated nonlinear wave models. Several applications of these equations are given as demonstrations, including a new improved Boussinesq model with fully nonlinear characteristics. The depth integration should not imply wave models of finite depth as long as the integral value of the imposed vertical distribution function remains finite. For instance, the recent wave model of Nadaoka *et al*. (1997), which uses Equation (20), is a typical example that can produce nonlinear waves (including the second-order Stokes waves) on infinitely deep water.

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