CHAPTER 32

A Fully-Dispersive Nonlinear Wave Model and its Numerical Solutions

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Abstract

A set of fully-dispersive nonlinear wave equations is derived by introducing a velocity expression with a few vertical-dependence functions and then applying the Galerkin method, which provides an optimum combination of the vertical-dependence functions to express an arbitrary velocity field under wave motion. The obtained equations can describe nonlinear non-breaking waves under general conditions, such as nonlinear random waves with a wide-banded spectrum at an arbitrary depth including very shallow and far deep water depths. The single component forms of the new wave equations, one of which is referred to here as "*time-dependent nonlinear mild-slope equation*", are shown to produce various existing wave equations such as Boussinesq and mild-slope equations as their degenerate forms. Numerical examples with comparison to experimental data are given to demonstrate the validity of the present wave equations and their high performance in expressing not only wave profiles but also velocity fields.

INTRODUCTION

Although evolution of non-breaking waves is principally governed by their nonlinearity and dispersivity, there exist no wave equations which can express these two effects under general conditions. For example, the Boussinesq-type equations are *weakly* nonlinear-dispersive equations and can describe only shallow water waves. Although several successful attempts for extending their applicable range in relative water depth have been reported (Madsen, et al., 1991; Nwogu, 1993, etc.), even such an improved model cannot be relied on if the depth becomes comparable with the wave length or more.

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The mild-slope equation of Berkhoff (1972) has no restriction on depth; but it can be used only for linear monochromatic (and hence *non-dispersive*) waves. The time-dependent forms of the mild-slope equation (e.g., Smith and Sprinks, 1975) can describe the dispersive evolution of linear random waves; but their band-width of spectrum is restricted to be narrow. (In this sense, they may be called "*narrow-banded* mild-slope equations".)

To break through all these restrictions, in the present study, new fully-dispersive nonlinear wave equations have been developed. Unlike the Boussinesq equations, which are derived with an asymptotic expansion procedure, the new equations are obtained by introducing a velocity expression with a few vertical-dependence functions and then applying the Galerkin method, which provides an optimum combination of the vertical-dependence functions to express an arbitrary velocity field of waves. The derived equations can express nonlinear non-breaking waves under general conditions, such as nonlinear random waves with a *wide-banded* spectrum at an *arbitrary depth* including very shallow and far deep water depths.

In the following sections, the principal idea and derivation procedure of the new wave equations* as well as their simplified forms are presented with some numerical examples and their comparison to experimental data to demonstrate the validity of the equations and the performance especially in expressing velocity fields. Besides theoretical relationships of the present theory to various existing wave equations such as Boussinesq and mild-slope equations are also shown.

THEORY

Principal Idea :

Generally speaking, any mathematical procedure to obtain a water-wave equation is a conversion process from original basic equations defined in a 3-D (x,y,z) space to wave equations to be defined in a horizontal 2-D (x,y) space. For this conversion, we must introduce an assumption on the vertical dependence of the velocity field.

For example, the Boussinesq equations are obtained by introducing the following expression with polynomials of z on the velocity potential Φ (e.g., Mei, 1983):

$$\Phi(x,y,z,t)=\sum_{m=0}^{\infty}\Phi_m(x,y,t)(z+h)^m, \tag{1}$$

where h is the water depth and the vertical coordinate z is taken upward from the still water level. With the Laplace equation of Φ and the boundary condition at the horizontal bottom, the above equation may be expressed as

$$\Phi(x,y,z,t)=\Phi_0-\frac{(z+h)^2}{2!}\nabla^2\Phi_0+\frac{(z+h)^4}{4!}\nabla^2\nabla^2\Phi_0-\cdots, \tag{2}$$

where $\nabla=(\partial/\partial x,\ \partial/\partial y)$. Usually only the first two terms in the above equation are retained to derive the Boussinesq equations.

This procedure is a kind of asymptotic expansion of Φ around the long wave limit,

* The fundamental idea of the present theory and numerical examples only for linear random waves with wide-band spectrum have been given in Nadaoka and Nakagawa (1991, 1993a,b). The extension to nonlinear waves but in more complicated form of equations has been reported in Nadaoka and Nakagawa (1993c).

and hence the Boussinesq equations can be applied only to shallow water waves. This restriction is related to the fact that the asymptotic approximate form of eq.(2) is not enough to express a velocity field under deeper waves. This in turn suggests that derivation of new wave equations with much wider applicability may be achieved by providing a more reasonable way to express the vertical dependence of a velocity field for more general cases including random waves in deep water.

In the present study, the following assumption is introduced to express the horizontal velocity vector, $\boldsymbol{q} = (u,v)$:

$$\boldsymbol{q}(x,y,z,t) = \sum_{m=1}^{N} \boldsymbol{U}_m(x,y,t)F_m(z), \tag{3}$$

where

$$F_m(z) = \frac{\cosh k_m(h+z)}{\cosh k_m h}. \tag{4}$$

The choice of cosh functions in the above as the vertical-dependence functions is based on the general 2-D solution of Laplace equation of Φ on the horizontal bottom (e.g., Nadaoka and Hino, 1983),

$$\Phi(x,z,t) = \int_{-\infty}^{\infty} A(k,t)\frac{\cosh k(h+z)}{\cosh kh}\exp(ikx)dk, \tag{5}$$

where k is the wavenumber and $A(k,t)$ is a time-varying wavenumber spectrum. It should be noted that eq.(5) is valid also for nonlinear waves and hence the use of eq.(4) as the vertical-dependence function $F_m(z)$ is not restricted to linear waves.

In the discrete form of eq.(5),

$$\Phi(x,z,t) \cong \sum_{i=1}^{i\max} A(k_i,t)\exp(ik_i x)\Delta k\frac{\cosh k_i(h+z)}{\cosh k_i h}, \tag{6}$$

we need a large number of the spectral component $A(k_i,t)$ in case of broad-banded random waves. However this fact does not necessarily mean that N in eq.(3) should be a large number, in spite of the resemblance between eqs.(3) and (6). This is true if each function, $\cosh k_i(h+z)/\cosh k_i h$, in eq.(6) can be expressed by eq.(3) with a *few* prescribed $F_m(z)$.

Galerkin Expression of a Velocity Field :

To examine this, the following approximation has been attempted:

$$\frac{\cosh k(h+z)}{\cosh kh} \cong \sum_{m=1}^{N} Q_m F_m(z), \tag{7}$$

where k is an arbitrary wavenumber and $F_m(z)$ is as defined in (4). For this approximation we need a mathematical procedure to determine the unknown coefficients Q_m $(m = 1,\cdots,N)$. For this purpose, in the present study, the Galerkin method has been employed.

Figure 1 shows the results of the approximation for five values of kh, covering very shallow to deep water depths. The number of components in eq.(7), N, is only 4 in this case and the prescribed values of $k_m h$ for $F_m(z)$ are 1.6, 3.5, 6.0, 10.5, respectively. The fact that the remarkably good agreements between the exact and approximated values are obtained for any arbitrary kh means that the

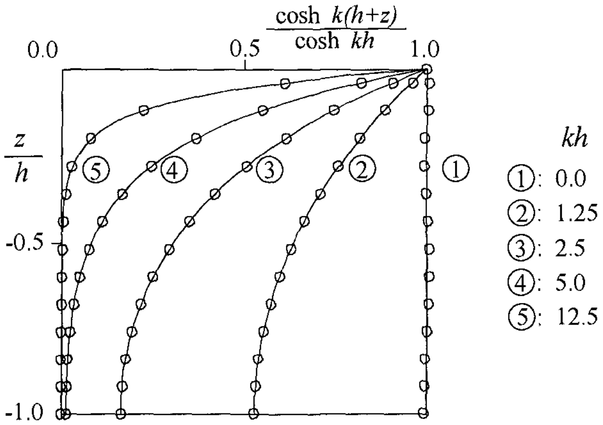


Fig. 1 Comparisons of the exact and approximated vertical distribution functions.

velocity expression by eqs.(3) and (4) with a small N may be applied to wave field under general conditions, such as random waves at an arbitrary depth including very shallow and far deep water depths. This is the most important finding to provide a basis of the new formulation of wave equations described in what follows.

Derivation of Fully-Dispersive Nonlinear Wave Equations :

With this basis of formulation, we are now ready to proceed to the derivation of new wave equations (for details, see Nadaoka et al., 1994).

The basic equations defined in 3-D (x,y,z) space are the continuity equation,

$$\nabla \cdot \boldsymbol{q} + \frac{\partial w}{\partial z} = 0, \tag{8}$$

and an alternative exact form of the Euler equation for irrotational flow (Beji,1994),

$$\frac{\partial \boldsymbol{q}}{\partial t} + \nabla\left[g\eta + \int_z^{\eta} \frac{\partial w}{\partial t} dz + \frac{1}{2}\left(\boldsymbol{q}_s \cdot \boldsymbol{q}_s + w_s^2 \right) \right] = 0, \tag{9}$$

where $\boldsymbol{q}_s$ and w_s are the velocity components at the free surface $z=\eta$.

The vertical velocity w is obtained from the continuity equation (8) by substituting eqs.(3) and (4) and integrating from the bottom to an arbitrary depth z:

$$w(x,y,z,t) = -\sum_{m=1}^{N} \nabla \cdot \left[\frac{\sinh k_m (h+z)}{k_m \cosh k_m h} \boldsymbol{U}_m(x,y,t) \right]. \tag{10}$$

The vertical integration of the continuity equation (8) over the entire depth gives

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left(\int_{-h}^{\eta} \boldsymbol{q} dz \right) = 0, \tag{11}$$

which then with the substitution of eqs.(3) and (4) yields

$$\frac{\partial \eta}{\partial t}+\sum_{m=1}^{N}\nabla\cdot\left[\frac{\sinh k_m(h+\eta)}{k_m\cosh k_m h}\boldsymbol{U}_m\right]=0. \tag{12}$$

To obtain the evolution equations of $\boldsymbol{U}_m$ $(m=1,\cdots,N)$, on the other hand, we may apply the Galerkin method to the momentum equation (9). Namely, after substituting eqs.(3)and (4) into eq.(9), the resulting equation is multiplied by the depth dependent function $F_m(z)$ and vertically integrated from z=-h to η. Since the depth-dependence function has N different modes, we obtain a total of N vector equations corresponding to each mode:

$$\sum_{m=1}^{N} a_{nm}\frac{\partial \boldsymbol{U}_m}{\partial t}+b_n\nabla\left[g\eta+\frac{1}{2}\left(\boldsymbol{q}_s\cdot\boldsymbol{q}_s+w_s^2\right)\right]=$$
$$\sum_{m=1}^{N}\left[c_{nm}\nabla\left(\nabla\cdot\boldsymbol{U}_m\right)_t+\boldsymbol{d}_{nm}\cdot\left(\nabla\cdot\boldsymbol{U}_m\right)_t\right], \qquad (n=1,2,\cdots,N) \tag{13}$$

where

$$a_{nm}=a_{mn}=\frac{1}{2\cosh k_m h\cosh k_n h}\left\{\frac{\sinh(k_m+k_n)(h+\eta)}{k_m+k_n}+\frac{\sinh(k_m-k_n)(h+\eta)}{k_m-k_n}\right\},$$

$$b_n=-g\frac{\sinh k_n(h+\eta)}{k_n\cosh k_n h},$$

$$c_{nm}=c_{mn}=\frac{1}{k_m^2\cosh k_m h\cosh k_n h}\left[\frac{\cosh k_m(h+\eta)\sinh k_n(h+\eta)}{k_n}\right.$$
$$\left.-\frac{1}{2}\left\{\frac{\sinh(k_m+k_n)(h+\eta)}{k_m+k_n}+\frac{\sinh(k_m-k_n)(h+\eta)}{k_m-k_n}\right\}\right]. \tag{14}$$

The coefficients $\boldsymbol{d}_{nm}$ in eq.(13) have rather complicated mathematical forms, but may be evaluated as being nearly equal to $\boldsymbol{D}_{nm}$ shown in eq.(19) later. In this evaluation the neglected terms are $O(\varepsilon\cdot\nabla h)$.

Equations (12) and (13) constitute a solvable set of equations for $2N+1$ unknowns, η, $\boldsymbol{U}_m$ $(m=1,\cdots,N)$, and describe their evolution as wave equations. It should be noted that no approximation has been introduced on the nonlinearity and that the full-dispersivity can be attained by taking only a few components, as demonstrated later; hence eqs.(12) and (13) may be referred to as "fully-dispersive nonlinear wave equations".

It should be further noted that k_m in eqs.(12) and (14) are *not* the wavenumbers in a usual sense like the spectral wavenumbers k_i in eq.(6), but they are the parameters to prescribe $F_m(z)$ so as to approximate a velocity field well enough. The wavenumber parameters k_m $(m=1,\cdots,N)$ are to be specified with the linear dispersion relation, $\omega_m{}^2= gk_m\tanh k_m h$, by prescribing the angular frequencies ω_m $(m=1,\cdots N)$ as a set of input data for the computation to properly eover the wave spectrum concerned. Therefore k_m must be treated as spatially varying quantities, according to the variation in $h(x,y)$.

Weakly Nonlinear Version of Fully-Dispersive Wave Equations :

Although equations (12) and (13) may express both full nonlinearity and dispersivity, they have disadvantages in computational aspects; i.e., the coefficients (14) includes many hyperbolic functions, besides the arguments of them have the unknown variable η. These points are undesirable in terms of computational time and robustness of the numerical algorithm.

Therefore, in the present study, a simplified version of eqs.(12) and (13) has been also developed by introducing a weakly-nonlinear formulation. By invoking a Taylor series expansion of $\boldsymbol{q}$ around z=0, and keeping only the first-order nonlinear contributions both in eqs.(9) and (11), we obtain

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left(\int_{-h}^{0} \boldsymbol{q} dz + \eta \boldsymbol{q}_0 \right) = 0, \tag{15}$$

$$\frac{\partial \boldsymbol{q}}{\partial t} + \nabla \left[g\eta + \int_{z}^{0} \frac{\partial w}{\partial t} dz + \eta \frac{\partial w_0}{\partial t} + \frac{1}{2}\left(\boldsymbol{q}_0 \cdot \boldsymbol{q}_0 + w_0^2 \right) \right] = 0, \tag{16}$$

in which $\boldsymbol{q}_0$ and w_0 are the velocities at the still water level z=0.

With the corresponding change of the upper limit of the vertical integration from η to 0 in the Galerkin procedure, we get the following simultaneous equations as the weakly-nonlinear version of eqs. (12) and (13).

$$\frac{\partial \eta}{\partial t} + \sum_{m=1}^{N} \nabla \cdot \left[\left(\frac{C_m^2}{g} + \eta \right) \boldsymbol{U}_m \right] = 0, \tag{17}$$

$$\sum_{m=1}^{N} A_{nm} \frac{\partial \boldsymbol{U}_m}{\partial t} + B_n \nabla \left[g\eta + \eta \frac{\partial w_0}{\partial t} + \frac{1}{2}\left(\boldsymbol{q}_0 \cdot \boldsymbol{q}_0 + w_0^2 \right) \right] =$$

$$\frac{\partial}{\partial t} \sum_{m=1}^{N} \left[C_{nm} \nabla (\nabla \cdot \boldsymbol{U}_m) + \boldsymbol{D}_{nm} (\nabla \cdot \boldsymbol{U}_m) \right], \quad (n = 1, 2, \cdots, N) \tag{18}$$

where,

$$A_{nm} = \frac{\omega_n^2 - \omega_m^2}{k_n^2 - k_m^2}, \qquad A_{nn} = \frac{g\omega_n^2 + h\left(g^2 k_n^2 - \omega_n^4\right)}{2gk_n^2}, \qquad \omega_m^2 = gk_m \tanh k_m h,$$

$$B_n = \frac{\omega_n^2}{k_n^2}, \qquad C_{nm} = \frac{B_n - A_{nm}}{k_m^2}, \qquad \boldsymbol{D}_{nn} = \nabla C_{nn}, \tag{19}$$

$$\boldsymbol{D}_{nm} = \frac{2}{k_m^2 - k_n^2} \left[\frac{2\nabla k_m}{k_m} \left\{ A_{nm} - \left(k_m^2 - k_n^2 \right) C_{nm} \right\} + \frac{g\nabla h}{\cosh k_n h \cdot \cosh k_m h} \right],$$

$\boldsymbol{q}_0$ and w_0 in eq.(18) may be evaluated as

$$\boldsymbol{q}_0 = \sum_{m=1}^{N} \boldsymbol{U}_m, \qquad w_0 = -\sum_{m=1}^{N} \nabla \cdot \left(\frac{B_m}{g} \boldsymbol{U}_m \right). \tag{20}$$

As shown in (19), the coefficients of the weakly nonlinear version of the equations are considerably simplified as compared with those defined in (14).

Linear Dispersion Characteristics of New Wave Equations :

The linear dispersion characteristics of the present wave equations can be examined by solving the eigenvalue problem defined with the linearized equation of the fully-dispersive equations and with the prescribed values of $k_m h$ (m=1,...,N). An example of the computed dispersion curve is shown in Fig. 2, where N=4 and the same values as those for Fig.1 are assigned to $k_m h$ (m=1,...,N). The computed values show perfect agreements with the theoretical linear dispersion curve over wide wavenumber domain extending from very shallow to far deep water. This remarkable feature of the present wave equations becomes more prominent by comparing with the dispersion curves of the classic Boussinesq equations and of the improved Boussinesq equations (Madsen et al. 1991), as shown in Fig.2.

The reason why the present equations can possess fully dispersive characteristics may be found by examining the dispersive characteristics of the linearized single-component (N=1) equation. In this case, the following analytical expression of the

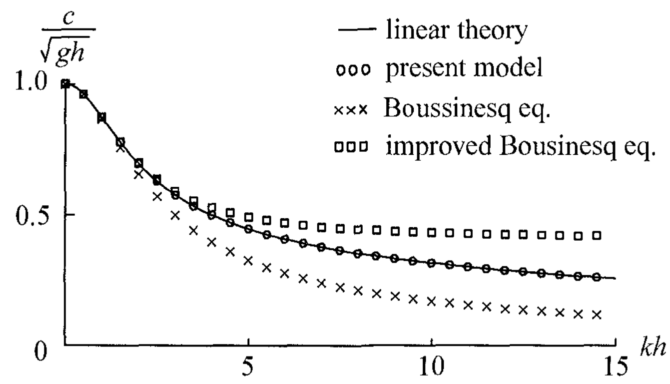


Fig.2 Linear dispersion characteristics of the fully-dispersive equations.

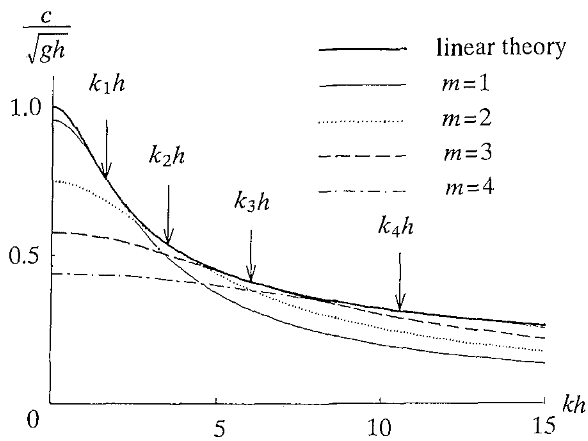


Fig.3 Dispersion curves of the single-component equation with each $k_i h$ for Fig.2.

dispersion relation can be obtained by solving the corresponding eigenvalue problem :

$$C^2 = \frac{C_p^3}{C_g + \frac{k^2}{k_p^2}\left(C_p - C_g\right)}, \tag{21}$$

where C_p and C_g are the theoretical linear phase and group velocities corresponding to the prescribed wavenumber k_p, while k and C denote an arbitrary incident wave-number and the corresponding phase celerity dictated by the dispersion relation (21). Figure 3 depicts the dispersion curves described by eq.(21), in which one of the values of $k_m h$ (m=1,...,N) for Fig.2 is given respectively for each curve as $k_p h$. As it is seen, each selected component describes a dispersion curve which is tangent to the exact curve at the selected wavenumber k_p. Hence combining all these contributions by the Galerkin procedure, we obtain the perfect agreement in the expression of the dispersive characteristics as shown in Fig.2.

Single-Component (N=1) Forms :

"*Narrow-banded* nonlinear wave equations"

The fact that as shown in Fig.3 each selected component describes a dispersion curve which is tangent to the exact curve at the selected wavenumber k_p means that if the waves in concern have a narrow-band spectrum centered at k_p, the single-component (N=1) versions of the wave equations (12) and (13) or (17) and (18) may be employed as "*narrow-banded* nonlinear wave equations".

For example, the single-component forms of eqs.(17) and (18) may be written as:

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left[\left(\frac{C_p^2}{g} + \eta \right) \boldsymbol{q}_0 \right] = 0, \tag{22}$$

$$C_p C_g \frac{\partial \boldsymbol{q}_0}{\partial t} + C_p^2 \nabla \left[g\eta + \eta \frac{\partial w_0}{\partial t} + \frac{1}{2}\left(\boldsymbol{q}_0 \cdot \boldsymbol{q}_0 + w_0^2 \right) \right] =$$

$$\frac{\partial}{\partial t} \left\{ \frac{C_p\left(C_p - C_g\right)}{k_p^2} \nabla(\nabla \cdot \boldsymbol{q}_0) + \nabla \left[\frac{C_p\left(C_p - C_g\right)}{k_p^2} \right] (\nabla \cdot \boldsymbol{q}_0) \right\}, \tag{23}$$

where C_p and C_g denote the phase and group velocities corresponding to k_p as defined by the linear theory.

By specifying C_p and C_g in these equations, we can show that various existing wave equations may be reproduced as the degenerate forms. For example, Airy's shallow water equations and Boussinesq equations can be obtained as follows.

(1) Airy's shallow water equations: $C_p = C_g = \sqrt{gh}$

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left[(h + \eta) \boldsymbol{q}_0 \right] = 0, \tag{24}$$

$$\frac{\partial \boldsymbol{q}_0}{\partial t} + \nabla \left(g\eta + \frac{1}{2} \boldsymbol{q}_0 \cdot \boldsymbol{q}_0 \right) = 0. \tag{25}$$

(2) Boussinesq equations:

$$C_p = \sqrt{gh}\left(1 - \frac{k_p^2 h^2}{6}\right), \qquad C_g = \sqrt{gh}\left(1 - \frac{k_p^2 h^2}{2}\right)$$

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left[(h+\eta)\boldsymbol{q}_0\right] + \frac{h^3}{3}\nabla^2(\nabla \cdot \boldsymbol{q}_0) = 0, \tag{26}$$

$$\frac{\partial \boldsymbol{q}_0}{\partial t} + \nabla\left(g\eta + \frac{1}{2}\boldsymbol{q}_0 \cdot \boldsymbol{q}_0\right) = 0, \tag{27}$$

where all the higher-order terms have been neglected.

Combined Form of the Single-Component Equations : *"Time-dependent nonlinear mild-slope equation"*

The single-component equations (22) and (23) may be combined, with the introduction of the mild-slope assumption, to give the following equation of η (Beji and Nadaoka, 1994):

$$C_g \eta_{tt} - C_p^3 \nabla^2 \eta - \frac{\left(C_p - C_g\right)}{k_p^2}\nabla^2 \eta_{tt} - C_p \nabla\left(C_p C_g\right) \cdot (\nabla \eta)$$
$$- \frac{3}{2} g C_p \left(3 - 2\frac{C_g}{C_p} - \frac{k_p^2 C_p^4}{g^2}\right)\nabla^2\left(\eta^2\right) = 0. \tag{28}$$

By further manipulations, the linearized equation of (28),

$$C_g \eta_{tt} - C_p^3 \nabla^2 \eta - \frac{\left(C_p - C_g\right)}{k_p^2}\nabla^2 \eta_{tt} - C_p \nabla\left(C_p C_g\right) \cdot (\nabla \eta) = 0, \tag{29}$$

can be found to lead to the time-dependent (or "*narrow-banded*") mild-slope equation proposed by Smith and Sprinks (1975),

$$\eta_{tt} + \omega_p^2 \left(\frac{C_p - C_g}{C_p}\right)\eta - \nabla\left(C_p C_g \nabla \eta\right) = 0, \qquad \left(\omega_p = C_p k_p\right) \tag{30}$$

and also to Berkhoff's (1972) elliptic equation as an original steady form of the mild-slope equation,

$$k_p^2 C_p C_g Z + \nabla \cdot \left(C_p C_g \nabla Z\right) = 0, \tag{31}$$

in which Z denotes a spatially varying wave amplitude. Therefore, eq.(28) can be regarded as an extension of the mild-slope equations to nonlinear waves. In this sense, eq.(28) may be called "*time-dependent nonlinear mild-slope equation*". However its linear dispersion characteristics are not the same as those of the time-dependent mild-slope equation (30), since the latter equation approximates more limited region around ω_p in the dispersion curve (Beji and Nadaoka, 1994). This means that even in the linear version of eq.(28), eq.(29), the new mild-slope equation has an advantage as compared with the conventional one.

Unidirectional Simplified Form of Nonlinear Mild-Slope Equation :

Equation (28) may be further elaborated for case of unidirectional propagation of waves in the positive x-direction only. The reason of taking up the analysis of such a simplified case lies in the attractive form of the KdV equation, which will be recovered as a special case. Skipping the derivation procedure (see Beji and Nadaoka, 1994) the equation we obtain is

$$C_g\eta_t + \frac{1}{2}C_p\left(C_p + C_g\right)\eta_x - \frac{\left(C_p - C_g\right)}{k_p^2}\eta_{xxt} - \frac{C_p\left(C_p - C_g\right)}{2k_p^2}\eta_{xxx}$$

$$+\frac{1}{2}\left[C_p\left(C_g\right)_x + \left(C_p - C_g\right)\left(C_p\right)_x\right]\eta + \frac{3}{4}g\left(3 - 2\frac{C_g}{C_p} - \frac{k_p^2C_p^4}{g^2}\right)\left(\eta^2\right)_x = 0, \qquad (32)$$

which describes the weakly-nonlinear wave evolution of a narrow-banded unidirectional wave field centered at the primary wave frequency $\omega_p = k_pC_p$.

The specification of C_p and C_g in eq.(32) yields again some degenerate forms. For weakly-dispersive shallow water waves, the specification,

$$C_p = \sqrt{gh}\left(1 - \frac{k_p^2h^2}{6}\right), \quad C_g = \sqrt{gh}\left(1 - \frac{k_p^2h^2}{2}\right)$$

leads to the KdV equation for a gently varying depth :

$$\eta_t + C_0\left[\eta_x + \frac{h_x}{4h}\eta + \frac{h^2}{6}\eta_{xxx} + \frac{3}{4h}\left(\eta^2\right)_x\right] = 0, \qquad (33)$$

in which $C_0 = \sqrt{gh}$.

For deep water, on the other hand, $C_p = \sqrt{g/k_p}$, $C_g = C_p/2$, then we have

$$\eta_t + \frac{3}{2}C_p\eta_x - \frac{1}{k_p^2}\eta_{xxt} - \frac{C_p}{2k_p^2}\eta_{xxx} + \frac{3}{2}\frac{g}{C_p}\left(\eta^2\right)_x = 0, \qquad (34)$$

which can be shown to admit the second-order Stokes' waves in deep water as an analytical solution.

NUMERICAL EXAMPLES

The forms of the single-component equations are in perfect correspondence with those of the Boussinesq equations. This is an important advantage because it allows the adoption of an efficient implicit scheme which has been developed for solving the Boussinesq equations. The numerical schemes for the combined forms of the single-component equations, (28) and (32), are of course much simpler and need shorter computational time.

The fully-dispersive equations, on the other hand, require a more complicated scheme to solve the N momentum equations. In the present study, a generalized Thomas algorithm, or the so-called block elimination method, is used for solving the linear algebraic equations resulting from an implicit three-time-level, centered discretization of the momentum equations (Nadaoka, et al., 1994). The values of

U_m $(m = 1,\cdots,N)$ at the boundary may be prescribed by applying a Galerkin procedure similar to that for eq.(7). The angular frequencies ω_m to specify the corresponding k_m $(m = 1,\cdots,N)$ are chosen so as to properly cover the frequency spectrum concerned. It has been found through various numerical computations, some of which will be shown later, that N required is usually no more than 3 and the use of slightly different set of ω_m $(m = 1,\cdots,N)$ yields no appreciable difference in the computational results and hence the apparent ambiguity in selecting ω_m $(m = 1,\cdots,N)$ does not affect the validity of the present model.

The following parts are devoted to show some typical numerical examples.

(1) Linear random waves

To examine the fundamental performance of the fully-dispersive equations, their linearized equations were applied to a case of linear random waves in deep water with a Bretschneider-type spectrum, which has a broad band-width in comparison with, e.g., JONSWAP. The relative water depth to the wave length corresponding to the mean period T_m is one (h/L_m=1). Figure 4 shows the comparisons with the predictions of linear theory for surface displacement and horizontal velocity at two different depths after 20 wave periods elapsed over a distance of five wavelengths. Good agreement with the theory is observed for both the surface displacement and velocity profiles. In the computation three components were used: $k_1h=2\pi$, $k_2h=3\pi$, $k_3h=5\pi$ with $\Delta x=L_m/90$ and $\Delta t= T_m/90$. The relatively fine resolutions were deemed necessary for the accurate representation of higher frequency components with shorter wavelengths and periods. No sponge layer was needed to improve the absorption at the outgoing boundary; the computational domain was not longer than shown. Good absorption of the radiating waves is attributable to the fact that the outgoing waves are radiated at three different wavenumbers instead of one. This is an important advantage especially in long time simulation of random waves.

(2) Nonlinear regular and irregular waves propagating over a bar

Further examinations on the fully-dispersive equations have been made through the comparisons with the laboratory data obtained by the experiment on the nonlinear wave deformation over a submerged trapezoidal bar as shown in Fig.5, which is similar to that of Beji and Battjes (1994). The two-component form of eqs. (17) and (18) was used for the computation by selecting the corresponding angular frequencies as $\omega_1=2\pi/T$ and $\omega_2=4\pi/T$. The experimental data compared is that for which the incident wave height H and period T are 2.0cm and 1.5s, respectively. Note that in this case the largest relative depth h/L observed was 0.35 at most. Figure 6(a) shows the comparisons for the water surface profiles at station 3, 5 and 7, while Fig.6(b) represents the velocity comparison at three depths at station 7, where an appreciable wave-decomposition phenomenon was observed. On the other hand, Figs.7(a) and (b) show the comparisons in which the improved Boussinesq equations of Madsen et al. (1991) were used for the computation. From these results, it is found that in the water surface profiles the present wave equations show good but nearly the same degree of agreements as compared with the improved Boussinesq equations. In the velocity profiles, on the contrary, the agreements for the improved Boussinesq equations deteriorate, although for the present equations the agreements are comparable to those in the surfaee profiles.

As a test for nonlinear random waves traveling over a submerged trapezoidal bar, the experimental data of Beji and Battjes (1994) was compared with the computational results by the two-component wave equations (17) and (18). The incident wave field has a JONSWAP type random wave spectrum with a peak period T_p=2s. The first four stations are in the upslope region where the nonlinear shoaling takes place. The remaining three stations are in the downslope region, where harmonic wave decomposition becomes appreciable. In the computations k_1 and k_2 for each component are selected to be k_p and πk_p, respectively, where k_p denotes the wavenumber corresponding to the peak period T_p. Figure 8 shows the results of the comparison at six different stations, indicating good agreements at all the stations. (It has been also found that even in case of using the simplest single-component equation (32) for the computations the agreements are slightly worse but still comparable to those indicated in Fig.8.)

(3) Stokes and cnoidal waves

The comparisons with the theories of steady nonlinear wave train have been also conducted by using the various versions of the present wave equations. The wave theories compared are of Stokes, cnoidal and solitary waves. As an example, Fig.9 show the comparisons with the second-order Stokes and cnoidal wave theories. The computations were carried out with the unidirectional simplified form of the nonlinear mild-slope equation (32). It is found that even this simplest version of equations can describe steady nonlinear wave trains with remarkably good accuracy under wide conditions including very shallow and far deep water waves. Solitary waves are also found to be well predicted by the present models, although the results are not presented here (see Nadaoka, et al.,1994; Beji & Nadaoka, 1994).

CONCLUSIONS

The major conclusions of the present study are summarized as follows :

1. Fully-dispersive nonlinear wave equations are presented which can express nonlinear non-breaking waves under general conditions, such as nonlinear random waves with *wide-spectrum* at an *arbitrary depth* including very shallow and far deep water depths.

2. The single-component forms of the new wave equations, one of which is referred to as "*time-dependent mild-slope equation*", are shown to produce various existing wave equations like Boussinesq and mild-slope equations as their degenerate forms.

3. Even under relatively shallow wave condition, present wave model can evaluate more precisely the velocity field than the improved Boussinesq equations.

Acknowledgement

The authors would like to thank Prof. J.A. Battjes for his permission to use the experimental data and a graduate student O. Ohno for his helpful cooperation in laboratory works.

References

Beji, S.(1994): Note on conservation equations for nonlinear surface waves. *Submitted to Coastal Eng.*

Beji, S. and Battjes, JA.(1994): Numerical simulation of nonlinear wave propagation over a bar, Coastal Eng., Vol.23, pp.1-16.

Beji, S. and Nadaoka, K.(1994): A time-dependent nonlinear mild-slope equation. *Submitted to Proc. Roy. Soc. London A*

Berkhoff, J.C.W.(1972): Computation of combined refraction-diffraction. Proc. 13th Int. Conf. Coastal Eng., ASCE, pp.471-490.

Madsen, P.A., Murray, R. and Sørensen, O.R.(1991): A new form of the Boussinesq equations with improved linear dispersion characteristics, Coastal Eng., Vol.15, pp. 371-388.

Mei, C.C.(1983): The applied dynamics of ocean surface waves, John Wiley & Sons, 740pp.

Nadaoka, K. and Hino, H.(1984): Conformal mapping solution of a wave field on the arbitrary shaped sea bottom, Proc. 19th Int. Conf. Coastal Eng., ASCE, pp.1192-1208.

Nadaoka, K and Nakagawa, Y.(1991): A Galerkin derivation of a fully-dispersive wave equation and its background, Tech. Rep. No.44, Dept. Civil Eng., Tokyo Inst. of Tech., pp.63-75. *(in Japanese)*

Nadaoka, K. and Nakagawa Y.(1993a): Fully-dispersive wave equations derived by a Galerkin formulation, *Meet'n '93*, ASCE/ASME/SES Abstracts, p.724.

Nadaoka, K. and Nakagawa, Y.(1993b): Derivation of fully-dispersive wave equations for irregular wave simulation and their fundamental characteristics, J. Hydraulics, Coastal and Environmental Eng., JSCE, No.467, pp.83-92. *(in Japanese)*

Nadaoka, K. and Nakagawa, Y.(1993c): Simulation of nonlinear wave fields with the newly derived nonlinear dispersive wave equations, Proc. Coastal Eng., JSCE, Vol.40, pp.6-10. *(in Japanese)*

Nadaoka, K., Beji, S. and Nakagawa, S.(1994): A fully-dispersive weakly-nonlinear wave model. *Submitted to Proc. Roy. Soc. London A*

Nwogu, O.(1993): Alternative form of Boussinesq equations for nearshore wave propagation, J. Waterway, Port, Coastal, and Ocean Eng., ASCE, Vol.119, No.6, pp.618-638.

Smith, R. and Sprinks, T.(1975): Scattering of surface waves by a conical island, J. Fluid Mech., Vol.72, pp.373-384.

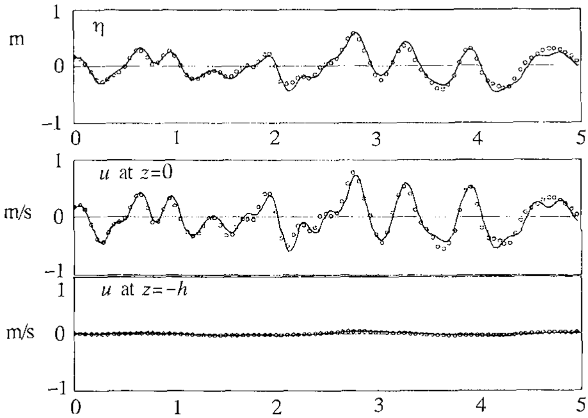


Fig.4 Linear random wave simulation ; linear theory (—) vs. computational results (O).

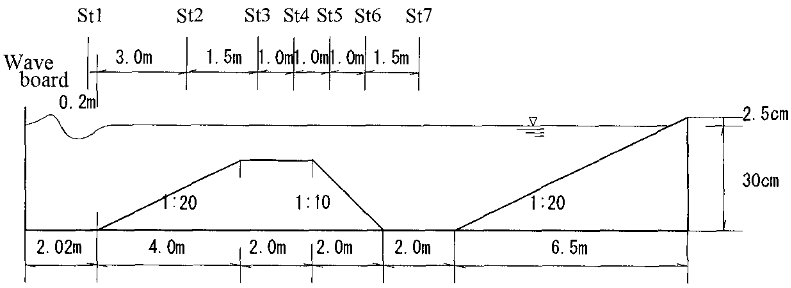
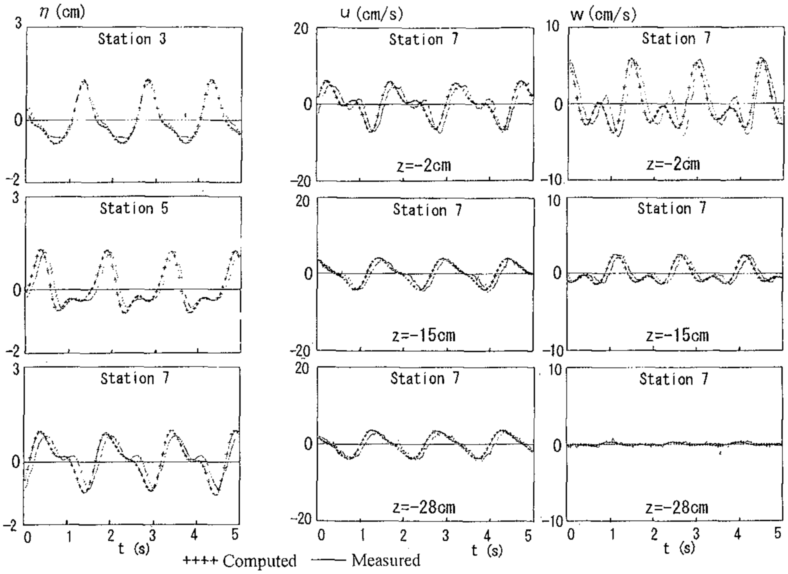
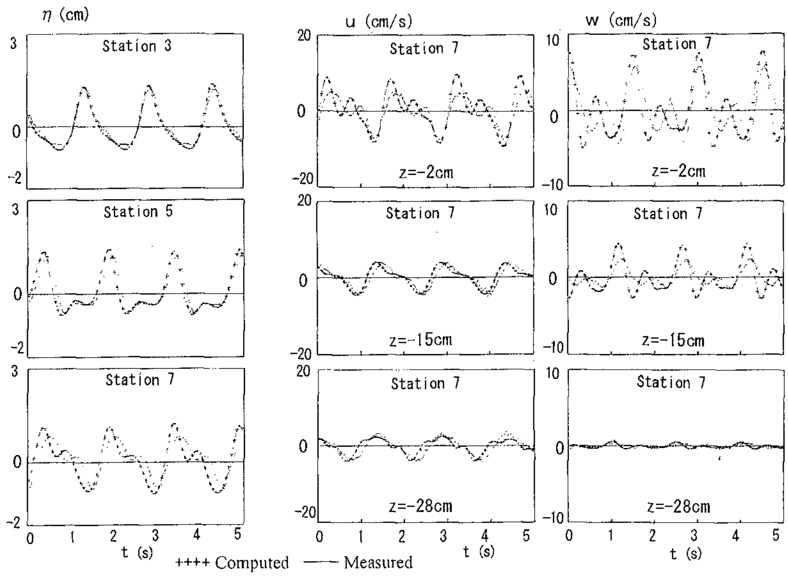


Fig.5 Definition sketch of wave flume and locations of wave gauges.



(a) η (b) u and w

Fig.6 Comparisons in η, u and w for the present fully-dispersive equations.



(a) η (b) u and w

Fig.7 Comparisons in η, u and w for the improved Boussinesq equations.

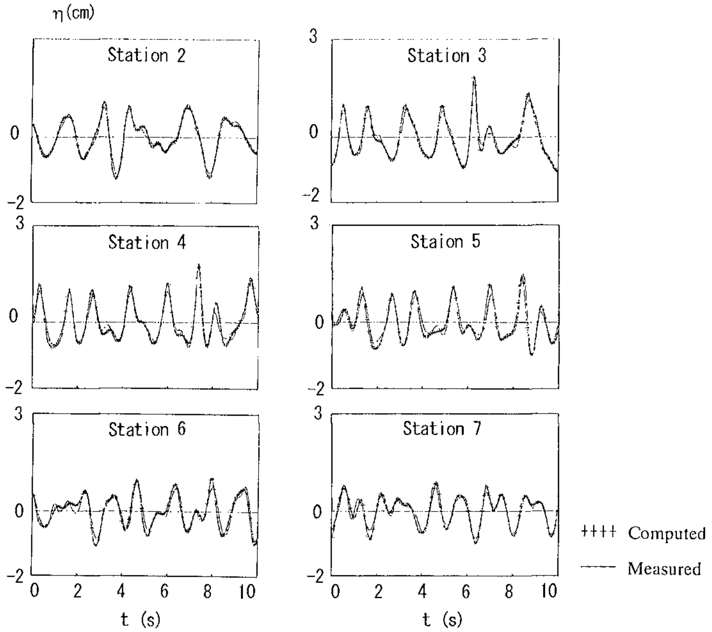
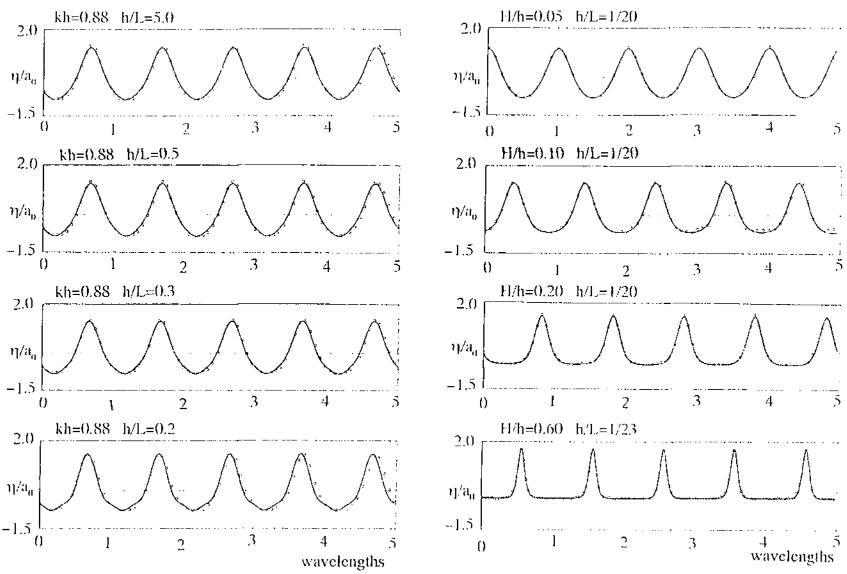


Fig.8 Comparison with laboratory data for nonlinear random waves propagating over a bar.



(a) Stokes waves (b) cnoidal waves

Fig.9 Comparison with Stokes and cnoidal wave theories; theory (—) vs. computed (O).