

PHYSICAL OCEANOGRAPHY

Density is a function of pressure, temperature, and salinity : $\rho = \rho(p, T, S)$. In oceanography usually g/cm^3 is used as the density unit.

Salinity : The amount of dissolved substance in grams in a 1 kg sea water.

35.00 ‰ (35 gr salt in 1000 gr seawater) 35.00 ppt (parts per thousand)

- If temperature increases density decreases $T \uparrow, \rho \downarrow$
- If salinity increases density increases $S \uparrow, \rho \uparrow$
- If pressure increases density increases $p \uparrow, \rho \uparrow$

In oceanography for computational ease density is defined as

$$\sigma = \rho - 1000 \quad (\text{kg/m}^3) \quad \text{or} \quad \sigma = \frac{(\rho - 1) \times 1000}{\uparrow \text{gr/cm}^3} \quad (\text{kg/m}^3)$$

For instance if $\rho = 1029.31 \text{ kg/m}^3$ or $\rho = 1.02931 \text{ gr/cm}^3$ then

$$\sigma = 1029.31 - 1000 = 29.31 \text{ kg/m}^3 \quad \text{or} \quad \sigma = (1.02931 - 1) \times 1000 = 29.31 \text{ kg/m}^3.$$

When in situ density ρ_{STP} is used for computing σ , it is labeled as σ_{stp} which indicates the density value at the salinity, temperature, and pressure measured "in situ".

(2)

If the density measured at the surface is used, $p=0$,

$$\sigma_{s,t,0} = \sigma_t = (\rho_{s,t,0} - 1000)$$

Specific volume is defined as the reciprocal of the density

$$\alpha_{s,t,p} = \frac{1}{\rho_{s,t,p}} \quad (\text{m}^3/\text{kg})$$

On the surface, $p=0$,

$$\alpha_{s,t,0} = \frac{1}{\rho_{s,t,0}} \quad (\text{m}^3/\text{kg})$$

Specific volume is often broke down into "specific anomalies"

$$\alpha_{STP} = \alpha_{35,0,p} + \delta_T + \delta_s + \delta_{ST} + \delta_{sp} + \delta_{TP} + \delta_{STP}$$

↓ ↓ ↓ ↓ ↓ ↓ ↑
 Temperature Salinity Salinity & Temperature Smaller than the rest
 correction correction & Temperature combined than the rest
 correction

Δ_{ST} (Thermosteric anomaly)

Almost always neglected

A useful empirical formula

$$\sigma_t = -0.0105 \Delta_{ST} + 28.1$$

$$\Delta_{ST} = -95.1 \sigma_t + 2675 \times 10^{-8} \text{ m}^3/\text{kg}$$

$$\alpha_{STP} = \alpha_{35,0,p} + \underbrace{\Delta_{ST}}_{\substack{\text{First obtain} \\ \sigma_t \text{ then}}} + \underbrace{\delta_{sp} + \delta_{TP}}_{\substack{\text{From tables} \\ \text{either read from tables or use empirical formula}}}$$

Computing the "in situ" density using tables

It can be shown that the pressure expressed in terms of "dbar" is nearly equal to the water depth in meters (as a quantity).

Pressure (db)	Temperature (°C)	Salinity (‰)	σ_t (kg/m³)	$10^5 \Delta_{ST}$ (cm³/gr)	$10^5 \delta_{SP}$ (cm³/gr)	$10^5 \delta_{TP}$ (cm³/gr)	$\alpha_{35.0, P}$ (cm³/gr)	$\alpha_{S,T,P}$ (cm³/gr)	σ_{STP} (kg/cm³)	
0	16	36.00	26.55	149.4	0.0	0.0	149.4	0.97264	0.97413	26.56

- a) Read σ_t from tables II-1 by first choosing the appropriate group of tables for salinity (here the range 30.00‰ - 39.95‰) and then using the given temperature. If the given temperature is not in the table then use the nearest lower value. For the above 16°C we use $T = 15.96^\circ\text{C}$ in the table and read $\sigma_t = 21.94$. However this is not the final value of σ_t , it must be corrected for salinity. Taking the last three digits of the salinity and using the correction factor f given for $T = 15.96^\circ\text{C}$ we compute the corrected σ_t as
- $$\sigma_t = 21.94 + 6.00 \times 0.7680 = 26.548 \text{ which must be rounded off to four digits to } \sigma_t = 26.55 \text{ for later computations}$$

- b) Use Table II-2 to read Δ_{ST} using $\sigma_t = 26.55$:

$$\Delta_{ST} = 149.4$$

c) $10^5 \delta_{sp} = ?$ From Table II-3b (page 312) we read

$$10^5 \delta_{sp} = 0.0 \quad \text{for } S = 36.00\% \text{ and } p = 0 \text{ db.}$$

d) $10^5 \delta_{tp} = ?$ From Table II-3b (page 314) we interpolate

$$10^5 \delta_{tp} = 0.0 \quad \text{for } T = 16.00^\circ C \text{ and } p = 0 \text{ db.}$$

e) $10^5 \delta = 10^5 \Delta_{ST} + 10^5 \delta_{sp} + 10^5 \delta_{tp} = 149.4 + 0.0 + 0.0 = 149.4 \text{ (cm}^3/\text{gr})$

f) $\alpha_{35,0,p} = ?$ Function of "p" only. For $p = 0$ db, from Table II-3a (page 311)

$$\alpha_{35,0,p} = 0.97264$$

g) $\alpha_{s,T,p} = ? \quad \alpha_{STP} = \alpha_{35,0,p} + \delta = 0.97264 + 149.4 \times 10^{-5} = 0.97413 \text{ (cm}^3/\text{gr})$

h) $\rho_{STP} = \frac{1}{\alpha_{STP}} = \frac{1}{0.97413 \frac{\text{cm}^3}{\text{gr}}} = 1.02656 \text{ gr/cm}^3 = 1026.56 \text{ kg/m}^3$

$$\tau_{STP} = \rho_{STP} - 1000 = 1026.56 - 1000 = \underline{\underline{26.56 \text{ kg/m}^3}}$$

Example to multiple interpolation

Compute $10^5 \delta_{sp}$ for $S = 37.35\%$ and $p = 560$ db.

1) Compute $10^5 \delta_{sp}$ for $S = 37.35\%$ and $p = 500$ db

$$\frac{\delta_{sp} - 1.5}{2.2 - 1.5} = \frac{37.35 - 37.00}{38.00 - 37.00}, \quad \delta_{sp} = 1.745 \times 10^{-5}$$

2) Compute $10^5 \delta_{sp}$ for $S = 37.35\%$ and $p = 600$ db

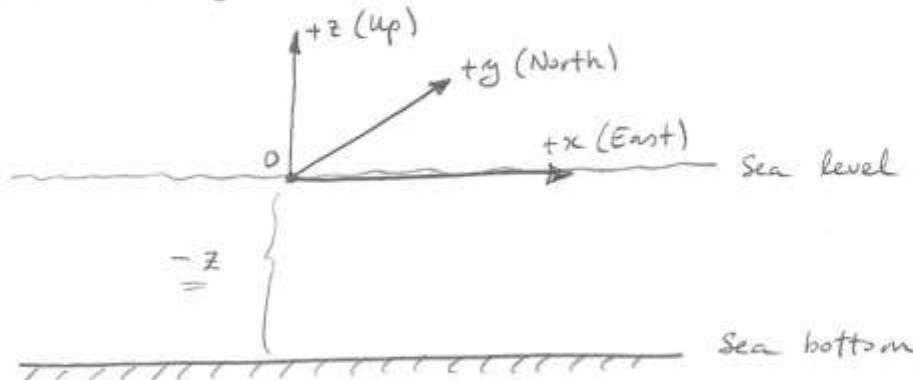
$$\frac{\delta_{sp} - 1.8}{2.6 - 1.8} = \frac{37.35 - 37.00}{38.00 - 37.00}, \quad \delta_{sp} = 2.08 \times 10^{-5}$$

3) Compute $10^5 \delta_{sp}$ for $p = 560$ db and $S = 37.35\%$

$$\frac{\delta_{sp} - 1.745}{2.08 - 1.745} = \frac{560 - 500}{600 - 500}, \quad \boxed{\delta_{sp} = 1.946 \times 10^{-5}}$$

Co-ordinate System and Advection

In oceanography the coordinate system used is located at the free surface :



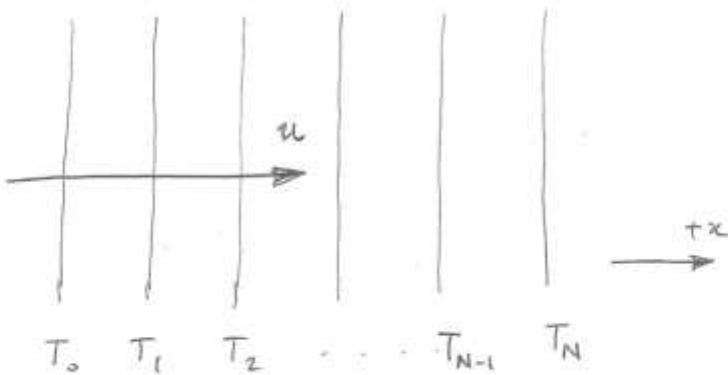
Physical meaning and derivation of "advection" terms : Consider a region with different temperatures and an object moving right with speed u .

T_0, T_1, \dots, T_N : Constant in time. Therefore,

$\frac{\partial T}{\partial t} = 0$. But, since

T_0, T_1, \dots, T_N are different

$$\frac{\partial T}{\partial x} \neq 0$$



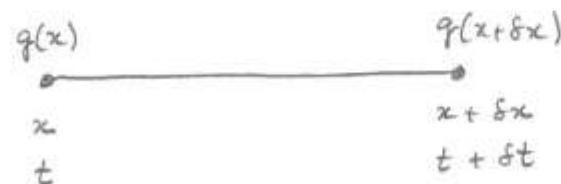
Although the temperature does not change in time, it changes spatially hence a particle moving to the right experiences a change of temperature in time due to its movement in a spatially non-zero temperature gradient.

Total change of temperature in time of particle

$$\frac{DT}{Dt} = \underbrace{\frac{\partial T}{\partial t}}_{\text{Local change of temperature}} + u \cdot \underbrace{\frac{\partial T}{\partial x}}_{\text{Advective change of temperature}}$$

(In one-dimensional case).

Let us now consider an arbitrary quantity q and derive its total change in time.



$$q(x+\delta x) = q(x) + \left(\frac{\partial q}{\partial x}\right) \cdot \delta x + \left(\frac{\partial^2 q}{\partial x^2}\right) \frac{\delta x^2}{2!} + \dots$$

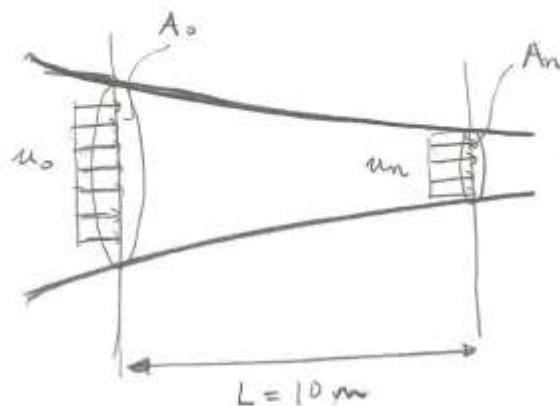
$$\frac{q(x+\delta x) - q(x)}{\delta t} = \left(\frac{\partial q}{\partial x}\right) \frac{\delta x}{\delta t} + \left(\frac{\partial^2 q}{\partial x^2}\right) \frac{\delta x}{2!} \cdot \frac{\delta x}{\delta t} + \dots$$

$$\lim_{\substack{\delta x \rightarrow 0 \\ \delta t \rightarrow 0}} \left[\frac{q(x+\delta x) - q(x)}{\delta t} \right] = \lim_{\substack{\delta x \rightarrow 0 \\ \delta t \rightarrow 0}} \left[\left(\frac{\partial q}{\partial x} \right) \frac{\delta x}{\delta t} + \underbrace{\left(\frac{\partial^2 q}{\partial x^2} \right) \frac{\delta x}{2!} \cdot \frac{\delta x}{\delta t}}_0 + \underbrace{\dots}_0 \text{ all the rest} \right]$$

$$\frac{Dq}{Dt} = u \left(\frac{\partial q}{\partial x} \right) \quad \text{for } q \text{ which is a function of "x" only.}$$

$$\frac{Dq}{Dt} = \left(\frac{\partial q}{\partial t} \right) + u \left(\frac{\partial q}{\partial x} \right) \quad \text{for } q \text{ which is a function of "x" and "t".}$$

Example to field acceleration: Consider a steady-state converging pipe-flow.



From the continuity of flow

$$u_o \cdot A_o = u_n \cdot A_n$$

$$u_n = u_o \left(\frac{A_o}{A_n} \right) \text{ since } A_o > A_n, u_n > u_o$$

The velocity changes spatially: $\frac{\partial u}{\partial x} \neq 0$ whereas $\frac{\partial u}{\partial t} = 0$

In one dimension we have

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \quad (\text{when } \eta \text{ is set to } u)$$

$$\frac{\partial u}{\partial t} = 0 \quad \text{but since } \frac{\partial u}{\partial x} \neq 0, \quad \frac{Du}{Dt} = u \frac{\partial u}{\partial x} \neq 0$$

In three dimensions

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$$

$$\text{For the velocity vector } \vec{V} = \vec{V}(u, v, w) = u \vec{i} + v \vec{j} + w \vec{k},$$

$$\frac{D\vec{V}}{Dt} = \frac{Du}{Dt} \vec{i} + \frac{Dv}{Dt} \vec{j} + \frac{Dw}{Dt} \vec{k} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k}$$

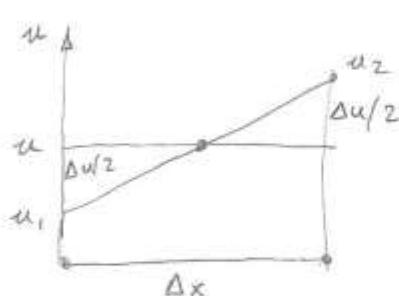
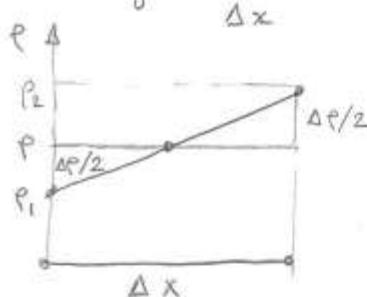
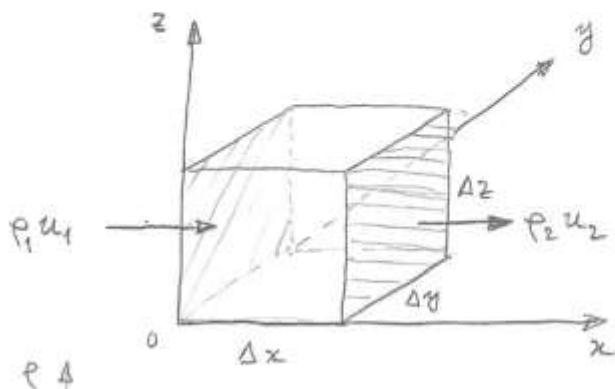
$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$$

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}$$

$$a_z = \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}$$

BASIC EQUATIONS : EQN OF CONTINUITY & MOMENTUM EQNS

(1) EQUATION OF CONTINUITY (CONSERVATION OF MASS)



Supposing that the mean density inside the cube whose sides are Δx , Δy , and Δz is ρ , then the mass inside the cube can be written as

$$m = \rho \Delta x \Delta y \Delta z$$

The change of mass m in time must be equal to the difference between the incoming flux and outgoing flux:

$$\frac{\partial m}{\partial t} = p_1 u_1 \Delta y \Delta z - p_2 u_2 \Delta y \Delta z$$

$$\frac{\partial}{\partial t} (\rho \Delta x \Delta y \Delta z) = p_1 u_1 \Delta y \Delta z - p_2 u_2 \Delta y \Delta z$$

$$\frac{\partial \rho}{\partial t} = \frac{p_1 u_1}{\Delta x} - \frac{p_2 u_2}{\Delta x} \quad \text{since } \Delta x, \Delta y, \text{ and } \Delta z \text{ are independent of time.}$$

$$\text{Assume } p_1 = \rho - \Delta p/2, \quad p_2 = \rho + \Delta p/2$$

$$u_1 = u - \Delta u/2, \quad u_2 = u + \Delta u/2$$

$$\frac{\partial \rho}{\partial t} = \frac{1}{\Delta x} \left[(\rho - \Delta p/2)(u - \Delta u/2) - (\rho + \Delta p/2)(u + \Delta u/2) \right]$$

(9)

After some algebra $\frac{\partial \rho}{\partial t} = \frac{1}{\Delta x} (-u \Delta \rho - \rho \Delta u)$, in the limit

$$\frac{\partial \rho}{\partial t} = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta \rho \rightarrow 0 \\ \Delta u \rightarrow 0}} \left(-u \frac{\Delta \rho}{\Delta x} - \rho \frac{\Delta u}{\Delta x} \right), \quad \frac{\partial \rho}{\partial t} = -u \frac{\partial \rho}{\partial x} - \rho \frac{\partial u}{\partial x} \quad (\text{For 1-D case})$$

If we generalize the formulation to the 3-D case

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w)$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0}$$

Continuity equation in its most general form.

using the definition of "total derivative" or "material derivative"

$$\frac{D\rho}{Dt} \triangleq \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \quad \text{we can express the above eqn as}$$

$$\boxed{\frac{D\rho}{Dt} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0}$$

If we assume $\rho = \text{constant}$ then $D\rho/Dt = 0$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

Equation of continuity for constant density

$$\boxed{\vec{\nabla} \cdot \vec{V} = 0}$$

$$\vec{\nabla} = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}$$

$$\vec{V} = u \vec{i} + v \vec{j} + w \vec{k}$$

An Application of the Equation of Continuity

We are going to use the continuity eqn for constant ρ :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{to compute}$$

the vertical velocity gradient and subsequently the vertical velocity itself:

$$\frac{\partial w}{\partial z} = - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

For finite-difference approximations we can write

$$\frac{\Delta w}{\Delta z} = - \left(\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} \right)$$

$\frac{\partial u}{\partial x}$ at point A:

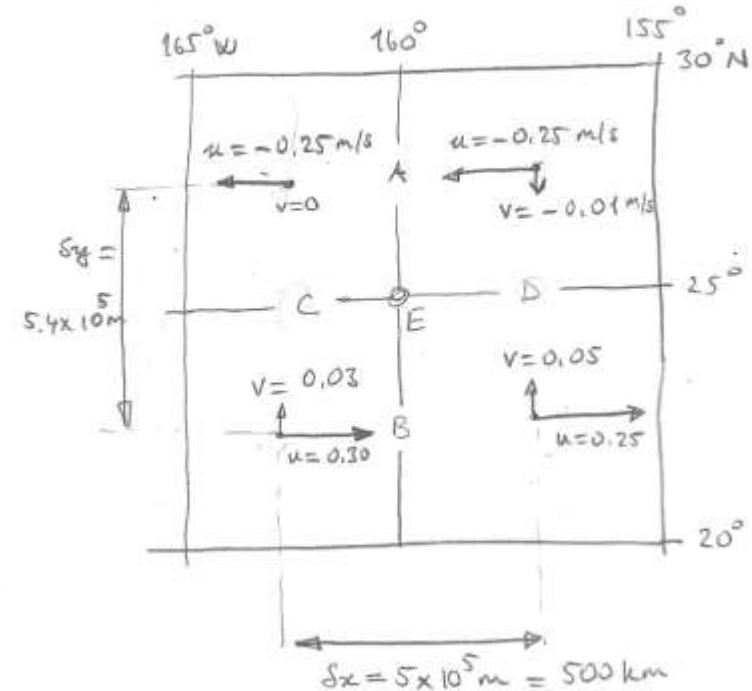
$$\left(\frac{\partial u}{\partial x} \right)_A \approx \left(\frac{\Delta u}{\Delta x} \right)_A = \frac{-0.25 - (-0.25)}{5 \times 10^5} = 0$$

$\frac{\partial u}{\partial x}$ at point B:

$$\left(\frac{\partial u}{\partial x} \right)_B \approx \left(\frac{\Delta u}{\Delta x} \right)_B = \frac{+0.25 - (+0.30)}{5 \times 10^5} = -10 \times 10^{-8} \text{ s}^{-1}$$

$\frac{\partial u}{\partial x}$ at point E:

$$\left(\frac{\partial u}{\partial x} \right)_E \approx \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)_A + \left(\frac{\partial u}{\partial x} \right)_B \right] = -5 \times 10^{-8} \text{ s}^{-1}$$



(11)

 $\frac{\partial v}{\partial y}$ at point C :

$$\left(\frac{\partial v}{\partial y}\right)_C \approx \left(\frac{\Delta v}{\Delta y}\right)_C = \frac{0 - (+0.03)}{5.4 \times 10^5} = -5.55 \times 10^{-8} \text{ s}^{-1}$$

 $\frac{\partial v}{\partial y}$ at point D :

$$\left(\frac{\partial v}{\partial y}\right)_D \approx \left(\frac{\Delta v}{\Delta y}\right)_D = \frac{-0.01 - (+0.05)}{5.4 \times 10^5} = -11.11 \times 10^{-8} \text{ s}^{-1}$$

 $\frac{\partial v}{\partial y}$ at point E :

$$\left(\frac{\partial v}{\partial y}\right)_E \approx \frac{1}{2} \left[\left(\frac{\partial v}{\partial y}\right)_C + \left(\frac{\partial v}{\partial y}\right)_D \right] \approx -8.3 \times 10^{-8} \text{ s}^{-1}$$

$$\left(\frac{\partial w}{\partial z}\right)_E = - \left[\left(\frac{\partial u}{\partial x}\right)_E + \left(\frac{\partial v}{\partial y}\right)_E \right] = - (-5 \times 10^{-8} - 8.3 \times 10^{-8}) = +13.3 \times 10^{-8} \text{ s}^{-1}$$

If the vertical velocity gradient $\left(\frac{\partial w}{\partial z}\right) > 0$ the vertical velocity gets smaller with increasing depth
 If the vertical velocity gradient $\left(\frac{\partial w}{\partial z}\right) < 0$ the vertical velocity gets larger with increasing depth

$$\frac{\partial w}{\partial z} = - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \quad \int_{-h}^0 \frac{\partial w}{\partial z} dz = - \int_{-h}^0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz$$

$$w(0) - w(-h) = - \int_{-h}^0 \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] dz = - \left[\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} \right] \Delta z \Big|_{-h}^0 = - \left[\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} \right] (0 + h)$$

$w(-h) = h \left[\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} \right]$

Let's compute the vertical velocity at $z = -50\text{ m}$ or $h = +50\text{ m}$.

$$w(-50) = 50 \times (-13.3 \times 10^{-8} \text{ s}^{-1}) = -6.7 \times 10^{-6} \text{ m/s} \approx -0.58 \text{ m/day} \text{ (Downwelling)}$$

In order to compute the time necessary for a particle to travel a given depth we first consider the definition of w :

$$w = \frac{dz}{dt}, \quad dt = \frac{dz}{w}. \quad \text{Using the previous formula } w(z) = -z \left[\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} \right]$$

$$\int_{t_1}^{t_2} dt = - \int_{z_1}^{z_2} \frac{dz}{z \left[\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta y} \right]} = - \frac{1}{-13.3 \times 10^{-8} \text{ s}^{-1}} \cdot \int_{z_1}^{z_2} \frac{dz}{z} = +7.5 \times 10^6 \text{ (s)} \int_{z_1}^{z_2} \frac{dz}{z} = +7.5 \times 10^6 \ln z \Big|_{z_1}^{z_2}$$

$$\Delta t = t_2 - t_1 = 7.5 \times 10^6 \ln \left(\frac{z_2}{z_1} \right)^{(s)} = 87 \ln \left(\frac{z_2}{z_1} \right) \text{ (days)}$$

$$\text{Let } z_1 = -1\text{ m}, \quad z_2 = -50\text{ m}$$

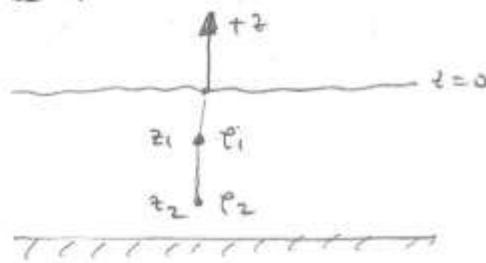
$$\Delta t = 87 \ln \left(\frac{-50}{-1} \right) = 87 \ln 50 \approx 340 \text{ days} \approx \underline{1 \text{ year}}$$

The time necessary for a particle at $z = -1\text{ m}$ to go down to $z_2 = -50\text{ m}$.

STABILITY - STATIC STABILITY

(13)

Here we consider whether or not the variation of density with depth is likely to cause the water move vertically. In a very simple way we can say that if $\frac{\partial \rho}{\partial z} < 0$ (density increases with depth) the fluid is statically stable and if $\frac{\partial \rho}{\partial z} > 0$ unstable.

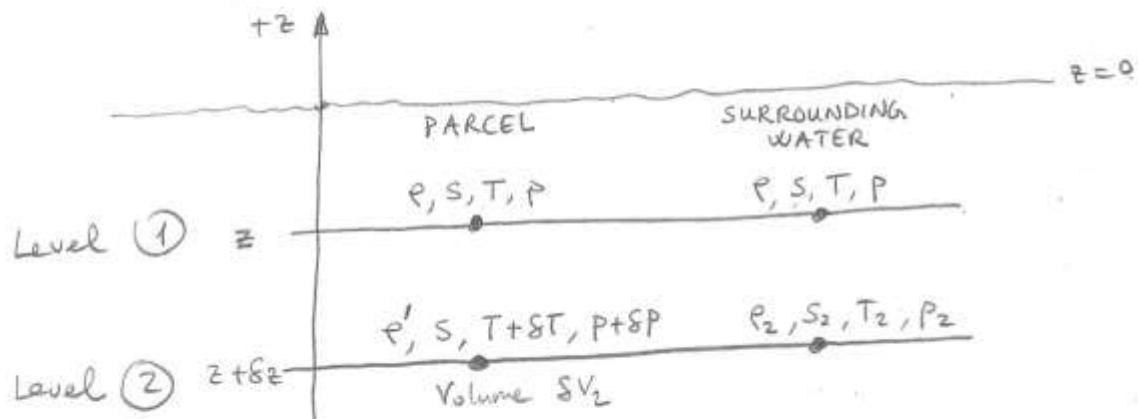


$$\frac{\partial \rho}{\partial z} \approx \frac{\rho_1 - \rho_2}{z_1 - z_2}$$

Always positive
when z_2 located
below z_1

Then, $\frac{\partial \rho}{\partial z} < 0$ if $\rho_2 > \rho_1$; $\frac{\partial \rho}{\partial z} > 0$ if $\rho_1 > \rho_2$

Derivation of the criterion for static stability



The volume of the parcel at level 2 is δV_2 .

We consider a parcel of water with density ρ , salinity s , temperature T and pressure p at level 1. We move the parcel down to level 2 whose surrounding water has density ρ_2 , salinity s_2 , temperature T_2 , and pressure p_2 .

The change in temperature due to the change in pressure can be written as

$$\delta T = \left(\frac{\partial T}{\partial p} \right)_{ad.} \delta p$$

adiabatic: without exchange of heat with the surrounding fluid.

By applying Archimedes' principle to the fluid parcel we can write for the buoyancy force F :

$$F = \underbrace{\delta V_2 \rho_2 g}_{\text{buoyant upthrust}} - \underbrace{\delta V_2 \rho' g}_{\text{weight}}$$

$$F = \delta V_2 g (\rho_2 - \rho')$$

Considering the corresponding acceleration gives

$$a_z = \frac{F}{M} = \frac{\delta V_2 g (\rho_2 - \rho')}{\delta V_2 \rho'} = \frac{g (\rho_2 - \rho')}{\rho' + \left(\frac{\partial \rho}{\partial z} \delta z \right)_p}$$

$$\left\{ \begin{array}{l} \rho_2 = \rho + \left(\frac{\partial \rho}{\partial z} \delta z \right)_W \\ \rho' = \rho + \left(\frac{\partial \rho}{\partial z} \delta z \right)_P \end{array} \right.$$

We are considering the change in density ρ at level 1 when a parcel of water is brought down to level 2 as $\rho + \left(\frac{\partial \rho}{\partial z} \delta z \right)_P$. Also, we are considering the change in density characteristics as we examine the surrounding water at level 1 and level 2.

$$\left[\frac{\partial \rho}{\partial z} \delta z \right]_W = \left[\frac{\partial \rho}{\partial S} \frac{\partial S}{\partial z} + \frac{\partial \rho}{\partial T} \frac{\partial T}{\partial z} + \frac{\partial \rho}{\partial P} \frac{\partial P}{\partial z} \right]_W \delta z, \quad \left[\frac{\partial \rho}{\partial z} \delta z \right]_P = \left[-\frac{\partial \rho}{\partial T} \Gamma + \frac{\partial \rho}{\partial P} \frac{\partial P}{\partial z} \right]_P \delta z$$

Since the salinity does not change with pressure when the parcel is moved down to level 2.

As we move down from level 1 to level 2 ($\frac{dp}{dz}$) value remains the same both for the surrounding water and the parcel. $(\frac{dp}{dz})_w = (\frac{dp}{dz})_p$

Similarly, $(\frac{\partial p}{\partial z})_w = (\frac{\partial p}{\partial z})_p$ as we move the same distance.

$$\frac{a_z}{g} = \frac{\left[\frac{\partial p}{\partial S} \cdot \frac{\partial S}{\partial z} + \frac{\partial p}{\partial T} \cdot \frac{\partial T}{\partial z} \right]_{\text{at}} \delta z + \left[\frac{\partial p}{\partial P} \cdot \frac{\partial P}{\partial z} \right]_{\text{at}} - \left[-\frac{\partial p}{\partial T} \Gamma \right]_{\text{at}} \delta z - \left[\frac{\partial p}{\partial P} \cdot \frac{\partial P}{\partial T} \right]_{\text{at}} \delta z}{\rho + \left(\frac{\partial p}{\partial z} \delta z \right)_p}$$

$$\frac{a_z}{g} = \frac{1}{\rho} \left[\frac{\partial p}{\partial S} \cdot \frac{\partial S}{\partial z} + \frac{\partial p}{\partial T} \left(\frac{\partial T}{\partial z} + \Gamma \right) \right] \delta z \quad \text{where } \frac{\partial p}{\partial z} \delta z \text{ in the denominator has been neglected in comparison with } \rho.$$

We define the stability E of the water column as $E = -\frac{a_z}{g \delta z}$ (- since a_z is upward which is already in the negative 'z' direction)

$$E = -\frac{1}{\rho} \left[\frac{\partial p}{\partial S} \cdot \frac{\partial S}{\partial z} + \frac{\partial p}{\partial T} \left(\frac{\partial T}{\partial z} + \Gamma \right) \right] \text{ (m}^{-1}\text{)} \quad \text{in which } \delta z = \text{unit length.}$$

For practical computations of stability the above eqn is not appropriate. It is possible to introduce a very simple criterion after considerable simplifications. It can be stated as

$$E = -\frac{1}{\rho} \cdot \frac{\partial \Gamma_t}{\partial z} \quad \text{stable if } E > 0$$

A better criterion which accounts for compressibility is given by

$$E = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial z} - \frac{g}{c^2} \quad \text{stable if } E > 0$$

where C is the speed of sound.

Example : Check if the given water column is stable.

$$p = 500 \text{ db}, \quad T = 16^\circ\text{C}, \quad S = 36.00\% \rightarrow \bar{\sigma}_t = 26.55 \text{ kg/m}^3 \text{ (from tables)}$$

$$p = 1000 \text{ db}, \quad T = 8^\circ\text{C}, \quad S = 36.00\% \rightarrow \bar{\sigma}_t = 28.09 \text{ kg/m}^3 \text{ (from tables)}$$

Using tables we compute $\rho_{st,p_{500}} = 1028.75 \text{ kg/m}^3$, $\rho_{st,p_{1000}} = 1032.61 \text{ kg/m}^3$

$$\bar{\rho} = \frac{1}{2} (\rho_{st,p_{500}} + \rho_{st,p_{1000}}) = \frac{1}{2} (1028.75 + 1032.61) = 1030.68 \text{ kg/m}^3$$

$$E = -\frac{1}{\rho} \frac{\partial \bar{\sigma}_t}{\partial z} = -\frac{1}{1030.68} \left(\frac{28.09 - 26.55}{-1000 - (-500)} \right) = -\frac{1}{1030.68} \times \frac{1.54}{-500} = +3 \times 10^{-6} \text{ m}^4 > 0 \text{ Stable}$$

$$c = 1449.22 + \Delta c_t + \Delta c_p + \Delta c_{st} + \Delta c_{stp}$$

$$c_{500} = 1449.22 + 61.12 + 8.05 + 1.31 - 0.264 = 1519.9 \text{ m/s}$$

$$c_{1000} = 1449.22 + 33.64 + 16.16 + 1.31 - 0.24 = 1500.1 \text{ m/s}$$

$$\bar{c}_{stp} = \frac{1}{2} (c_{500} + c_{1000}) = \frac{1}{2} (1519.9 + 1500.1) = 1509.8 \text{ m/s.}$$

$$E = -\frac{1}{\rho} \frac{\partial \bar{c}}{\partial z} - \frac{\partial}{c^2} = -\frac{1}{1030.68} \left(\frac{1032.61 - 1028.75}{-1000 - (-500)} \right) - \frac{9.81}{1509.8^2} = 7.5 \times 10^{-6} - 4.3 \times 10^{-6} = +3.2 \times 10^{-6} \text{ m}^4 > 0 \text{ Stable.}$$

WATER, SALT AND HEAT BUDGETS OF THE OCEANS

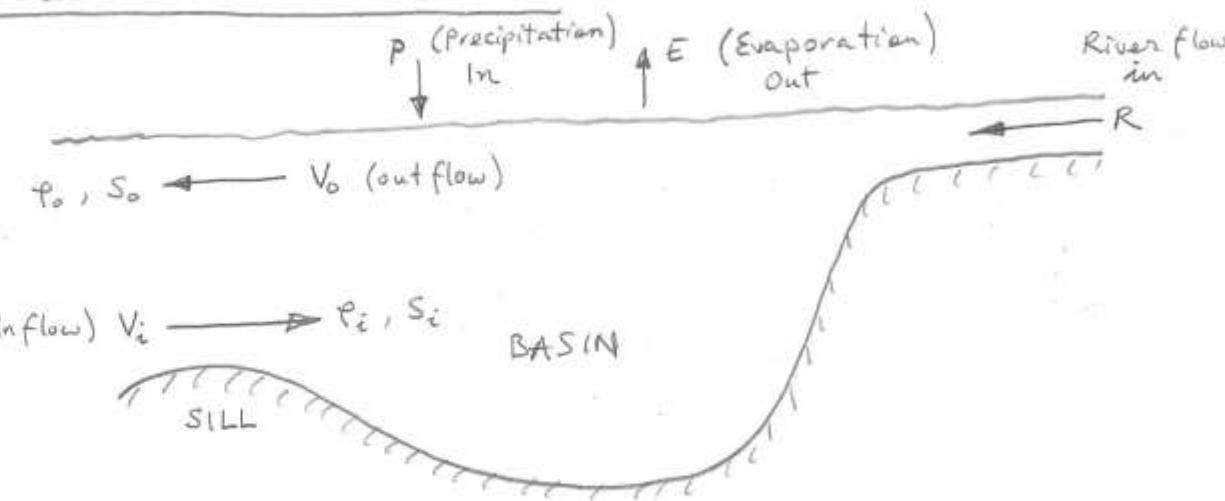
1) CONSERVATION OF VOLUME

We can state the conservation of volume for the basin as

$$V_i + R + P = V_o + E \quad \text{OR}$$

$$V_o - V_i = (R + P) - E = X$$

V: Volume transport (m^3/s)



2) CONSERVATION OF SALT

The principle of conservation of salt states that the amount of salt in the oceans remains very nearly constant therefore symbolically we write

$$V_i \cdot \rho_i \cdot S_i = V_o \cdot \rho_o \cdot S_o$$

where S_i and S_o are the salinities respectively of the inflowing and the outflowing sea-water. Since the difference between the ocean and fresh water densities would be at most 3% we can drop ρ_i and ρ_o and write

$$V_i \cdot S_i = V_o \cdot S_o$$

as the conservation of salt. Combining the above relation with the conservation of volume we get Knudsen's relations $V_i = X \cdot S_o / (S_i - S_o)$ and $V_o = X \cdot S_i / (S_i - S_o)$.

The Mediterranean Sea

Direct measurements of the upper layer currents

give an average value for $V_i = 1.75 \times 10^6 \text{ m}^3/\text{s}$.

From the conservation of salt $V_o = V_i S_i / S_o$

$V_o = 1.68 \times 10^6 \text{ m}^3/\text{s}$. Using the conservation of

volume $X = (R+P) - E = V_o - V_i = -7 \times 10^4 \text{ m}^3/\text{s} < 0$

Evaporation exceeds fresh-water input by $7 \times 10^4 \text{ m}^3/\text{s}$.

$V_i = 1.75 \times 10^6 \text{ m}^3/\text{s} = 5.5 \times 10^4 \text{ km}^3/\text{year}$ at this rate, considering that the volume of Mediterr. basin as $3.8 \times 10^6 \text{ km}^3$, it would take 70 years to fill the Mediterranean. This may be taken roughly as a measure of the residence time (sometimes called flushing time).

The Black Sea

Measured values for $V_i = 6 \times 10^3 \text{ m}^3/\text{s}$. Again,

using the conservation of salt $V_o = 13 \times 10^3 \text{ m}^3/\text{s}$,

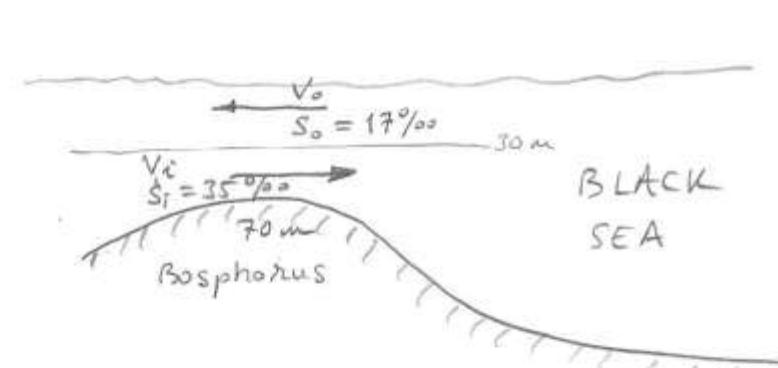
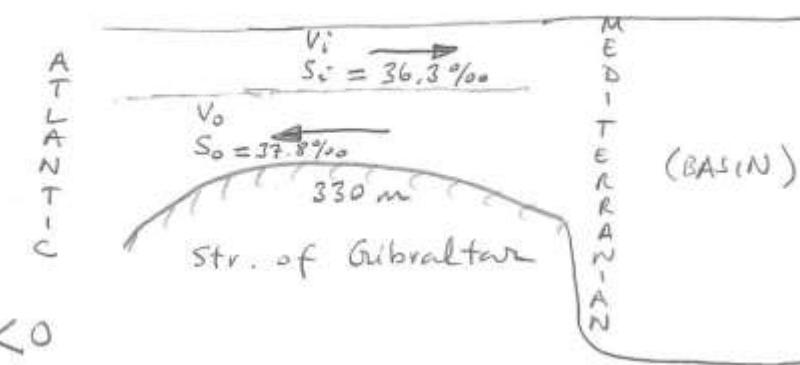
which gives $X = V_o - V_i = (R+P) - E = +6.5 \times 10^3 \text{ m}^3/\text{s}$

$X > 0$ there is a net inflow of fresh water

to the sea.

$V_i = 6 \times 10^3 \text{ m}^3/\text{s} = 0.02 \times 10^4 \text{ km}^3/\text{year}$. Compared

to the Black Sea volume of $0.6 \times 10^6 \text{ km}^3$, this suggests a residence time of 3000 years.



THE EQUATION OF MOTION IN OCEANOGRAPHY

We consider Newton's second law of motion $\vec{F} = m\vec{a}$, $\vec{a} = \frac{\vec{F}}{m}$: Force per unit mass

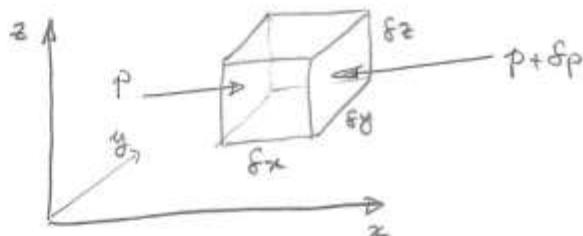
Recall that $\vec{a} = \frac{d\vec{V}}{dt} = \underbrace{\left(\frac{\partial \vec{V}}{\partial t} \right)}_{\text{Local acceleration}} + \underbrace{(\vec{V} \cdot \vec{\nabla}) \cdot \vec{V}}_{\text{Field acceleration}}$

$$\underbrace{u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}}$$

Total acceleration

$$\vec{a} = \frac{\vec{F}}{m} \Rightarrow \frac{d\vec{V}}{dt} + (\vec{V} \cdot \vec{\nabla}) \cdot \vec{V} = \frac{\vec{F}}{m} = \text{Pressure Force} + \text{Gravity (only in the } z\text{-direction)} + \text{Coriolis Force} + \text{Other Forces}$$

Pressure Force



The pressure force acting on the left side
+ $p \delta y \delta z$

The pressure force acting on the right side
- $(p + \delta p) \delta y \delta z$

The net pressure in the x -direction:

$$\vec{f}_x = \vec{i} \cdot p \cdot \delta y \delta z - \vec{i} \cdot (p + \delta p) \delta y \delta z = -i \delta p \delta y \delta z = -i \frac{\delta p}{\delta x} \cdot \delta x \delta y \delta z$$

The mass of the cube whose sides are δx , δy , and δz is $\delta m = \rho \delta x \delta y \delta z$

Then the net pressure force acting on per unit mass becomes

$$\vec{F}_x = \frac{\vec{f}_x}{\delta m} = \frac{-i \frac{\delta p}{\delta x} \cdot \delta x \delta y \delta z}{\rho \delta x \delta y \delta z} = -i \frac{1}{\rho} \frac{\delta p}{\delta x} \text{ in the limit} = -i \frac{1}{\rho} \frac{dp}{dx}$$

The pressure force per unit mass in the x -direction is then

$$-\frac{1}{\rho} \cdot \frac{\partial p}{\partial x} \vec{i} \quad \text{or} \quad -\alpha \frac{\partial p}{\partial x} \vec{i}$$

Similarly, we can write down the pressure forces per unit mass in the y - and z -directions as

$$-\alpha \frac{\partial p}{\partial y} \vec{j} \quad \text{and} \quad -\alpha \frac{\partial p}{\partial z} \vec{k}$$

In the vector notation the pressure force per unit mass is

$$-\alpha \vec{\nabla} p$$

Considering the total force per unit mass again

$$\vec{a} = \frac{\vec{F}}{m} = \underbrace{-\alpha \vec{\nabla} p}_{\text{pressure force}} - \underbrace{g \vec{k}}_{\substack{\text{gravitational} \\ \text{force}}} + \text{Coriolis} + \text{other forces}$$

$\vec{a} = \frac{d\vec{v}}{dt} + (\vec{v} \cdot \vec{\nabla}) \vec{v}$ in a fixed reference frame. If the acceleration is expressed

in a rotating reference frame (such as the earth) :

$$\vec{a}_f = \left(\frac{d\vec{v}'}{dt} \right)_f = \left(\frac{d\vec{v}'}{dt} \right)_e + 2\vec{\omega} \lambda \vec{v} + \vec{\omega} \lambda (\vec{\omega} \lambda \vec{R})$$

\vec{v}' : Velocity vector relative to the fixed axes

\vec{v} : Velocity vector relative to the earth

\vec{R} : the vector distance of the body from the center of the earth

$\vec{\omega}$: Angular velocity of rotation of the earth $|\vec{\omega}| = \omega = 7.29 \times 10^{-5} \frac{\text{rad}}{\text{s}}$

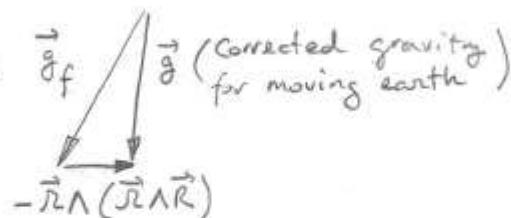
Sidereal : Of,
concerned with, or
measured by the stars:
sidereal time.
< Lat. Sidereus.

$$\left\{ \begin{array}{l} \text{One sidereal day} \\ = 23 \text{ h } 56 \text{ m } 4 \text{ s} \\ = 86164 \text{ s} \\ \omega = \frac{2\pi \text{ rad}}{86164 \text{ s}} = 7.29 \times 10^{-5} \text{ rad/s} \end{array} \right.$$



The second term $\vec{r} \wedge (\vec{r} \wedge \vec{R})$ causes a slight change in the gravitational acceleration

Gravity according to the fixed stars



The difference between $|\vec{g}_f|$ and $|\vec{g}|$ is only 0.5% therefore the term $\vec{r} \wedge (\vec{r} \wedge \vec{R})$ can be neglected.

After neglecting $\vec{r} \wedge (\vec{r} \wedge \vec{R})$ the acceleration in the moving reference system

$$\left(\frac{d\vec{v}}{dt} \right)_e = \left(\frac{d\vec{v}}{dt} \right)_f - 2\vec{r} \wedge \vec{v} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} - 2\vec{r} \wedge \vec{v}$$

Before we derived the pressure and other forces acting on a fluid particle, from Newton's second law of motion we can now write

$$\frac{d\vec{v}}{dt} = -\alpha \vec{\nabla} p - 2\vec{r} \wedge \vec{v} + \vec{g} + \vec{F}$$

as the equation of motion expressed in a co-ordinate system moving with the earth.

In terms of the components

$$\frac{du}{dt} = -\alpha \frac{\partial p}{\partial x} + 2r \sin \phi \cdot v - 2r \cos \phi \cancel{w} + F_x$$

$$\frac{dv}{dt} = -\alpha \frac{\partial p}{\partial y} - 2r \sin \phi \cdot u + F_y$$

Neglected because $w \ll g$

$$\frac{dw}{dt} = -\alpha \frac{\partial p}{\partial z} + 2r \cos \phi \cdot u - g + F_z$$

Neglected because $2r \cos \phi \cdot u \ll g$

We can state the equations of motion used in oceanography as



$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\alpha \frac{\partial p}{\partial x} + fv + F_x$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\alpha \frac{\partial p}{\partial y} - fu + F_y$$

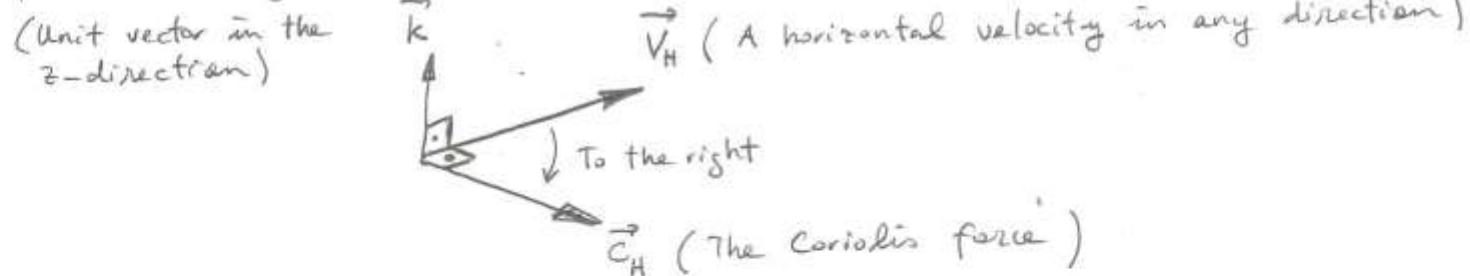
$$\alpha \frac{\partial p}{\partial z} = -g + F_z$$

where $f = 2R \sin \phi$
with ϕ being latitude angle

Note that ω is completely neglected as the pressure is assumed hydrostatic.

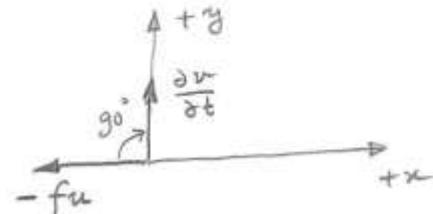
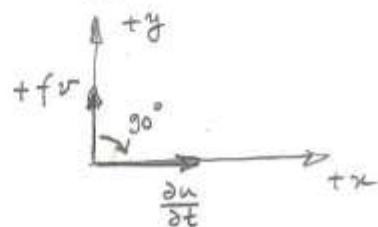
If we don't know the latitude angle ϕ we can approximately take $f = 10^{-5} \frac{\text{rad}}{\text{sec}}$

In the northern hemisphere the Coriolis force acts to the right (left in the southern hemisphere) of the moving particle:



From the above eqns in the simplest forms we have

$$\frac{\partial u}{\partial t} = +fv \quad \text{and} \quad \frac{\partial v}{\partial t} = -fu$$

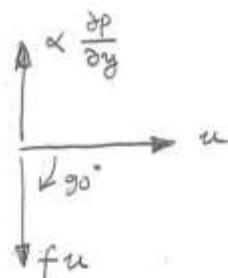


GEOSTROPHIC FLOW

We consider a nearly steady-state motion in which $\frac{du}{dt}$ and $\frac{dv}{dt}$ are negligibly small. Only the pressure gradient and the Coriolis force balance each other. x- and y-momentum eqns are then reduced to

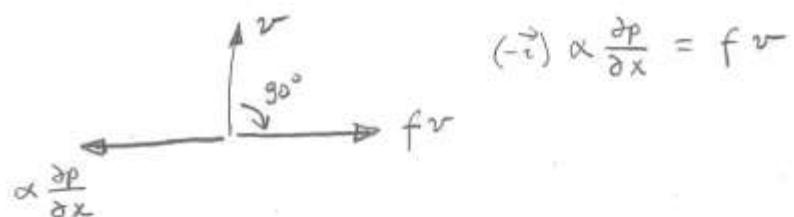
$$\left. \begin{aligned} 0 &= -\alpha \frac{\partial p}{\partial x} + fv \\ 0 &= -\alpha \frac{\partial p}{\partial y} - fu \end{aligned} \right\} \quad \text{Coriolis force} = -\text{Pressure force}$$

$$u \neq 0, v = 0$$



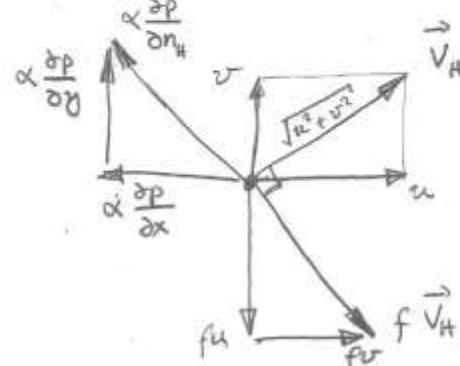
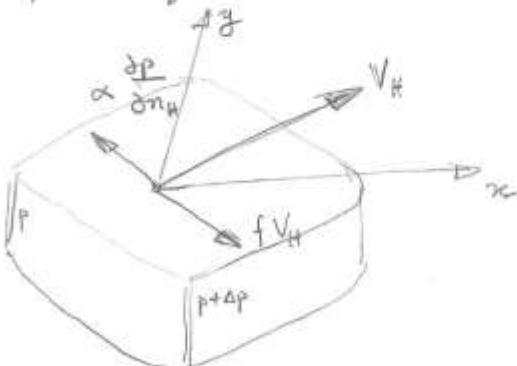
$$(+j) \cdot \alpha \frac{\partial p}{\partial y} = -fu$$

$$u = 0, v \neq 0$$



$$(-i) \alpha \frac{\partial p}{\partial x} = fv$$

If we generalize this idea, for any direction



$$0 = -\alpha \frac{\partial p}{\partial n_H} + f V_H$$

$$f V_H = \alpha \frac{\partial p}{\partial n_H}$$

Coriolis force = Pressure Force

The Geostrophic Method for Calculating Relative Velocities

$$\overline{AA_1} = z_2$$

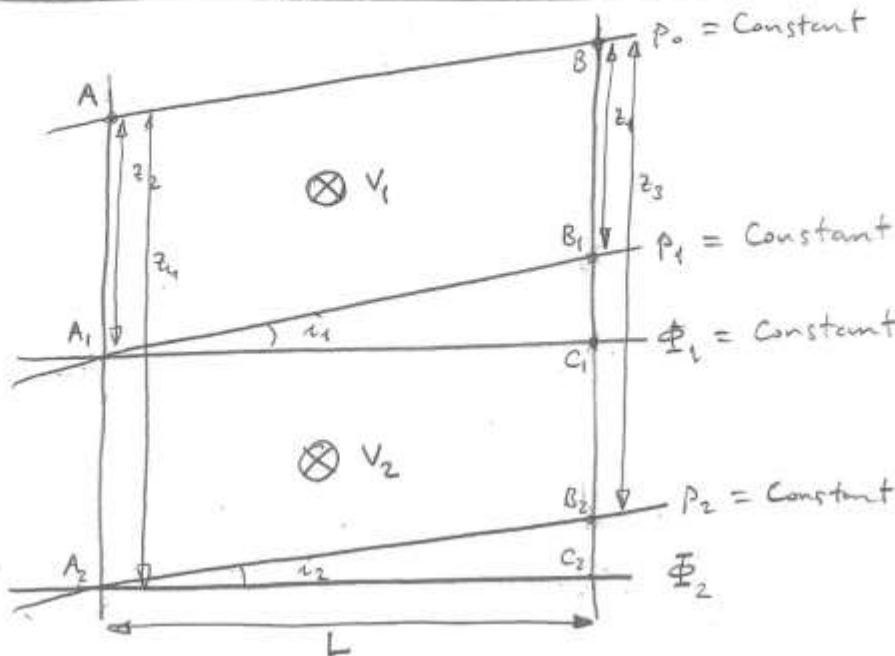
$$\overline{AA_2} = z_4$$

$$\overline{BB_1} = z_1$$

$$\overline{BB_2} = z_3$$

Geostrophic equation

$$fV = g \tan i$$



For the upper layer

$$fV_1 = g \tan i_1$$

For the lower layer

$$fV_2 = g \tan i_2$$

$$f(V_1 - V_2) = g(\tan i_1 - \tan i_2)$$

$$f(V_1 - V_2) = g \left(\frac{\overline{B_1 C_1}}{\overline{A_1 C_1}} - \frac{\overline{B_2 C_2}}{\overline{A_2 C_2}} \right) = \frac{g}{L} (\overline{B_1 C_1} - \overline{B_2 C_2}) = \frac{g}{L} (\overline{B_1 B_2} - \overline{C_1 C_2})$$

$$f(V_1 - V_2) = \frac{g}{L} (\overline{B_1 B_2} - \overline{A_1 A_2}) = \frac{g}{L} [(z_1 - z_3) - (z_2 - z_4)]$$

From the hydrostatic equation

$$\int_{B_1}^{B_2} g dz = - \int_{P_1}^{P_2} \alpha dp \Rightarrow g z \Big|_{z_1}^{z_3} = - \int_{P_1}^{P_2} \alpha dp$$

$$g(z_3 - z_1) = - \left[\int_{P_1}^{P_2} \alpha_{3S,0,p} dp + \int_{P_1}^{P_2} \delta_B dp \right] \quad \text{since } \alpha_{S+p} = \alpha_{3S,0,p} + \delta$$

Repeating the same approach for the station A we get

$$\int_{A_1}^{A_2} g dz = - \int_{P_1}^{P_2} \alpha dp, \quad g(z_4 - z_2) = - \left[\int_{P_1}^{P_2} \alpha_{3S,0,p} dp + \int_{P_1}^{P_2} \delta_A dp \right]$$

We multiply the both eqns by -1 and subtract the second eqn from the first

$$g[(z_1 - z_3) - (z_2 - z_4)] = \int_{P_1}^{P_2} \delta_B dp - \int_{P_1}^{P_2} \delta_A dp \quad \text{After making use of our previous relationship,}$$

$$f(v_1 - v_2) = \frac{1}{L} \left[\int_{P_1}^{P_2} (\delta_B - \delta_A) dp \right] = \frac{1}{L} (\Delta \Phi_B - \Delta \Phi_A)$$

Since, by definition, $\Delta \Phi_B \triangleq \int_{P_1}^{P_2} \delta_B dp$ and $\Delta \Phi_A \triangleq \int_{P_1}^{P_2} \delta_A dp$.

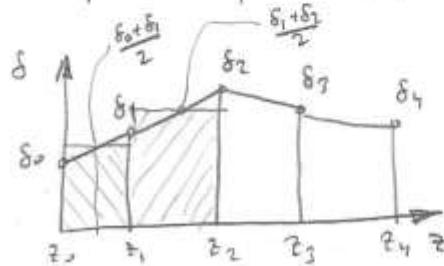
$$v_1 - v_2 = \frac{1}{L \cdot 2 \pi \sin \phi} \left[\int_{P_1}^{P_2} \delta_B dp - \int_{P_1}^{P_2} \delta_A dp \right]$$

$$v_1 - v_2 = \frac{1}{L \cdot 2 \pi \sin \phi} [\Delta \Phi_B - \Delta \Phi_A]$$

Example

Station A at $41^{\circ}55'N$ and $50^{\circ}09'W$ and Station B at $41^{\circ}28'$ and $50^{\circ}09'W$. For Station A measurements give

Depth (m)	T ($^{\circ}C$)	S (‰)	$\sigma_t \left(\frac{\text{kg}}{\text{m}^3} \right)$	$10^8 \Delta s_t \left(\frac{\text{m}^3}{\text{kg}} \right)$	$10^8 \delta_{s,p}$	$10^8 \delta_{t,p}$	$10^8 \delta$	$10^8 \bar{\delta}$	$\frac{\delta \times \Delta z}{10000}$	$\Delta \Phi_A = \sum (\delta \times \Delta p)$
0	5.99	33.71	26.56	148	0	0	148	146	0.365	6.638
25	6.00	33.78	26.61	144	0	1	144	135	0.338	6.273
50	10.30	34.86	26.81	125	0	2	126	122	0.315	5.935
75	10.30	34.88	26.83	123	0	2	119	112	0.305	5.620
100	10.10	34.92	26.89	117	0	3	104	99	0.455	5.315
150	10.25	35.17	27.06	101	0	4	93	83	0.830	4.700
200	8.85	35.03	27.19	89	0	5	73	65	0.650	3.470
300	6.85	34.93	27.41	68	0	5	57	52	1.040	2.820
400	5.55	34.93	27.58	52	0	7	46	45	0.900	1.780
600	4.55	34.95	27.71	39	0	8	45	44	0.880	0.880
800	4.25	34.95	27.74	37	0					0
1000	3.90	34.95	27.78	33	0	10	43			



$$\Delta t_1 = \left(\frac{\delta_0 + \delta_1}{2} \right) \times (z_1 - z_0)$$

$$\Delta t_2 = \left(\frac{\delta_1 + \delta_2}{2} \right) \times (z_2 - z_1)$$

etc.

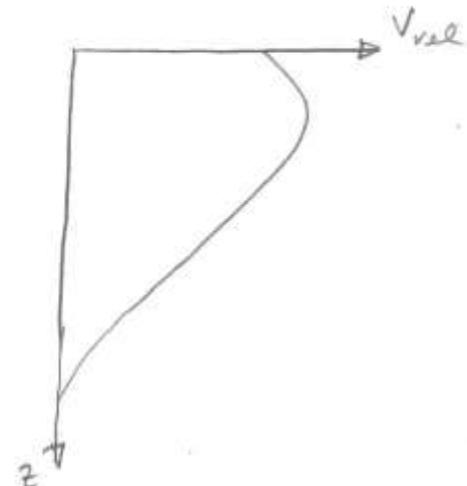
$$\int_{P_1}^{P_2} \delta \, dP = \bar{\delta} \Delta P = 10^4 \bar{\delta} \Delta z$$

$$P = P g z \approx 1000 \left(\frac{\text{kg}}{\text{m}^3} \right) \times 10 \left(\frac{\text{m}}{\text{s}^2} \right) \times z$$

$$P \approx 10^4 z \quad \text{or} \quad \Delta P = 10^4 \Delta z$$

The same computations are performed for the station B and then the following table is constructed

Depth (m)	$\Delta \Phi_B \left(\frac{m^2}{s^2} \right)$	$\Delta \Phi_A \left(\frac{m^2}{s^2} \right)$	$(\Delta \Phi_B - \Delta \Phi_A)$	$V_{rel} (m/s)$
0	7.894	6.638	1.256	0.26
25	7.596	6.293	1.323	0.27
50	7.298	5.935	1.363	0.28
75	7.000	5.620	1.380	0.29
100	6.700	5.315	1.385	0.29
150	6.090	4.755	1.335	0.28
200	5.480	4.300	1.180	0.24
300	4.310	3.470	0.840	0.17
400	3.330	2.820	0.510	0.11
600	1.930	1.780	0.150	0.03
800	0.900	0.880	0.020	0.005
1000	0	0	0	0



$$V_{rel} = V_1 - V_2 = \frac{1}{L \cdot 2R \sin \phi} [\Delta \Phi_B - \Delta \Phi_A]$$

St. A $41^\circ 55' 00'' N$

St. B $41^\circ 28' 00'' N$

$27' 00''$

$$L = 27 \times 1852 = 50000 \text{ m}$$

$$1' = 1 \text{ nautical mile} = 1852 \text{ m}$$

$$\bar{\phi} = \frac{41^\circ 55' + 41^\circ 28'}{2} = 41^\circ 41' 30'' = 41 + \frac{41}{60} + \frac{30}{3600}$$

$$\bar{\phi} = 41.69^\circ, \quad r = 7.29 \times 10^{-5} \frac{\text{rad}}{\text{sec}}, \quad L \cdot 2R \sin \phi = 50000 \times 2 \times 7.29 \times 10^{-5} \times \sin 41.69^\circ = 4.8486$$

EQUATIONS OF MOTION FOR THE MEAN OR AVERAGE FLOW

We let the velocity be expressed as a mean and fluctuating part

$$u = \bar{u} + u'$$

with the assumption that the average of u' over a definite period is zero. $\frac{1}{T} \int_0^T u' dt = 0$

Considering the local acceleration term in the momentum eqn:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} (\bar{u} + u') , \quad \frac{1}{T} \int_0^T \frac{\partial u}{\partial t} dt = \frac{1}{T} \int_0^T \frac{\partial}{\partial t} (\bar{u} + u') dt = \frac{\partial \bar{u}}{\partial t} + \overset{\uparrow}{\underset{\text{Mean}}{\int_0^T u' dt}} + \overset{\uparrow}{\underset{\text{Fluctuating}}{\int_0^T u'^2 dt}}$$

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{u}}{\partial t}$$

For pressure

$$\alpha \cdot \frac{\partial p}{\partial x} = (\bar{\alpha} + \alpha') \frac{\partial (\bar{p} + p')}{\partial x} = \bar{\alpha} \frac{\partial \bar{p}}{\partial x} + \bar{\alpha}' \frac{\partial p'}{\partial x} + \overset{\circ}{\bar{\alpha}'} \frac{\partial \bar{p}}{\partial x} + \overset{\circ}{\bar{\alpha}'} \frac{\partial p'}{\partial x} = \bar{\alpha} \frac{\partial \bar{p}}{\partial x}$$

If we consider the nonlinear terms such as

$$\begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= (\bar{u} + u') \frac{\partial (\bar{u} + u')}{\partial x} + (\bar{v} + v') \frac{\partial (\bar{u} + u')}{\partial y} + (\bar{w} + w') \frac{\partial (\bar{u} + u')}{\partial z} \\ &= \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} + \underbrace{u' \frac{\partial \bar{u}}{\partial x} + v' \frac{\partial \bar{u}}{\partial y} + w' \frac{\partial \bar{u}}{\partial z}}_{\text{Reynolds Stresses}} + \underbrace{u' \frac{\partial u'}{\partial x} + v' \frac{\partial u'}{\partial y} + w' \frac{\partial u'}{\partial z}}_{\text{Stresses}} \end{aligned}$$

Let's manipulate the Reynolds stresses as follows

$$\overline{u' \frac{\partial u}{\partial x} + v' \frac{\partial u}{\partial y} + w' \frac{\partial u}{\partial z}} + \underbrace{\overline{u'} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)}_{\text{Zero (due to the continuity)}}$$

Re-writing the above expression as

$$\frac{\partial}{\partial x} (\overline{u'u'}) + \frac{\partial}{\partial y} (\overline{u'v'}) + \frac{\partial}{\partial z} (\overline{u'w'})$$

Then, the x-momentum eqn for the mean motion as used in oceanography may be written as

$$\begin{aligned} \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} &= -\bar{\alpha} \frac{\partial \bar{p}}{\partial x} + f \bar{v} - \underbrace{\frac{\partial}{\partial x} (\overline{u'u'}) - \frac{\partial}{\partial y} (\overline{u'v'}) - \frac{\partial}{\partial z} (\overline{u'w'})}_{\text{Reynolds stresses}} \\ &+ \gamma \left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2} \right) \end{aligned}$$

Eddy or Turbulent Viscosity Concept

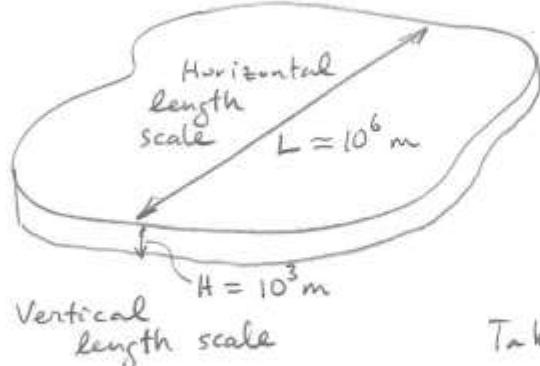
In order to put the Reynolds stresses into a manageable form we make the following assumptions :

$$-\overline{u'u'} = A_x \frac{\partial \bar{u}}{\partial x}, \quad -\overline{u'v'} = A_y \frac{\partial \bar{u}}{\partial y}, \quad -\overline{u'w'} = A_z \frac{\partial \bar{u}}{\partial z}$$

where A_x , A_y , and A_z are "eddy" or "turbulent" viscosity coefficients.

$$-\frac{\partial}{\partial x} (\overline{u'u'}) - \frac{\partial}{\partial y} (\overline{u'v'}) - \frac{\partial}{\partial z} (\overline{u'w'}) = \frac{\partial}{\partial x} (A_x \frac{\partial \bar{u}}{\partial x}) + \frac{\partial}{\partial y} (A_y \frac{\partial \bar{u}}{\partial y}) + \frac{\partial}{\partial z} (A_z \frac{\partial \bar{u}}{\partial z})$$

SCALING THE EQUATIONS OF MOTION ; ROSSBY NUMBER , EKMAN NUMBER



For the horizontal length scale in oceans we can take $L \sim 10^6 \text{ m}$ and the vertical length scale $H \sim 10^3 \text{ m}$.

Considering the continuity eqn $\frac{\partial w}{\partial t} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$

$$\frac{w}{H} \sim \frac{u}{L}, \quad w \sim \frac{H}{L} \cdot u$$

Taking typical horizontal speed $u \sim 0.1 \text{ m/s}$, $w \sim \frac{10^3}{10^6} \times 10^{-1} \sim 10^{-4} \text{ m/s}$
typical vertical speed $w \sim 10^{-4} \text{ m/s}$.

For a typical time scale T we take $10 \text{ days} = 10^6 \text{ s}$. For the Coriolis term f

we take $f = 2\Omega \sin \phi = 2 \times 7.29 \times 10^{-5} \times \sin 45^\circ \simeq 10^{-4} \text{ rad/s}$. For the pressure term
 $p = \rho g z = 10^3 \times 10 \times 10^3 = 10^7 \text{ pa}$. The eddy viscosity coefficients $A_x, A_y \sim 10^{-10^5} \text{ m}^2/\text{s}$
we take the extreme case $A_x, A_y \sim 10^5 \text{ m}^2/\text{s}$. $A_z \sim 10^{-5} - 10^{-1} \text{ m}^2/\text{s}$ we take $A_z \sim 10^{-1} \text{ m}^2/\text{s}$

Let us apply the above scaling procedure to the vertical momentum eqn first :

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\alpha \frac{\partial p}{\partial z} + fu - g + A_x \frac{\partial^2 w}{\partial x^2} + A_y \frac{\partial^2 w}{\partial y^2} + A_z \frac{\partial^2 w}{\partial z^2} \\ \frac{w}{T} + u \frac{w}{L} + v \frac{w}{L} + w \frac{w}{H} &= -\alpha \frac{p}{H} + fu - g + A_x \frac{w}{L^2} + A_y \frac{w}{L^2} + A_z \frac{w}{H^2} \\ \frac{10^{-4}}{10^6} + 10^{-1} \frac{10^{-4}}{10^6} + 10^{-1} \frac{10^{-4}}{10^6} + 10^{-4} \frac{10^{-7}}{10^3} &= + 10^{-3} \frac{10^7}{10^3} + 10^{-4} \frac{10^{-1}}{10} - 10 + 10^5 \frac{10^{-4}}{10^{12}} + 10^5 \frac{10^{-4}}{10^{12}} + 10^{-1} \frac{10^7}{10^6} \\ 10^{-10} + 10^{-11} + 10^{-11} + 10^{-11} &= \boxed{+ 10} + 10^{-5} \boxed{- 10} + 10^{-11} + 10^{-11} + 10^{-11} \end{aligned}$$

Therefore in the vertical momentum eqn it is quite sufficient to keep only the hydrostatic pressure term and the gravitational term

$$-\frac{1}{\rho} \cdot \frac{\partial p}{\partial z} - g = 0 , \quad \frac{\partial p}{\partial z} = -\rho g , \quad p = -\rho g z$$

Consider the horizontal momentum eqn in the x -direction:

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= -\alpha \frac{\partial p}{\partial x} + fv - fw + Ax \frac{\partial^2 u}{\partial x^2} + Ay \frac{\partial^2 u}{\partial y^2} + Az \frac{\partial^2 u}{\partial z^2} \\ \frac{u}{T} + u \frac{u}{L} + v \frac{u}{L} + w \frac{u}{H} &= -\alpha \frac{\partial p}{\partial x} + fv - fw + Ax \frac{u}{L^2} + Ay \frac{u}{L^2} + Az \frac{u}{H^2} \\ \frac{10^1}{10^6} + 10^1 \frac{10^1}{10^6} + 10^1 \frac{10^1}{10^6} + 10^{-4} \frac{10^{-1}}{10^3} &= -\alpha \frac{\partial p}{\partial x} + 10^{-4} 10^{-1} - 10^{-4} 10^{-1} + 10^5 \frac{10^1}{10^{12}} + 10^5 \frac{10^1}{10^{12}} + 10^1 \frac{10^1}{10^6} \\ 10^{-7} + 10^{-8} + 10^{-8} + 10^{-8} &= -\alpha \frac{\partial p}{\partial x} + 10^{-5} - 10^{-8} + 10^{-8} + 10^{-8} + 10^{-8} \end{aligned}$$

Multiplying the entire eqn by 10^5 we get

$$10^{-2} + 10^{-3} + 10^{-3} + 10^{-3} = \boxed{-10^5 \alpha \frac{\partial p}{\partial x} + 1} - 10^{-3} + 10^{-3} + 10^{-3} + 10^{-3}$$

Since the remaining terms are on the order of 10^{-2} or 10^{-3} we must have a balance between the pressure term $-\alpha \frac{\partial p}{\partial x}$ and the Coriolis term fv which gives the geostrophic balance

$$0 = -\alpha \frac{\partial p}{\partial x} + fv$$

We then conclude that for large scale oceanic motions we must use the geostrophic eqns:

$$\partial = -\alpha \frac{\partial p}{\partial x} + fv, \quad \partial = -\alpha \frac{\partial p}{\partial y} - fu, \quad \partial = -\alpha \frac{\partial p}{\partial z} - g$$

valid for the interior of the ocean a few degrees or more away from the equator.

On the other hand, for small scale motions $L \sim 10^3$ m the acceleration term can balance the Coriolis term:

Acceleration \sim Coriolis Term

$$\frac{u^2}{L} \sim fu, \quad \frac{(10^1)^2}{10^3} = \frac{10^{-4}}{10^{-1}} \Rightarrow 10^5 \sim 10^{-5}$$

To classify some other kind of motions we define the following non-dimensional numbers

Rossby Number $Ro = \frac{\text{Non-linear terms}}{\text{Coriolis term}} = \frac{u^2/L}{f_0 u} = \boxed{\frac{u}{f_0 L} = Ro}$ Important if $Ro \approx 1$

Ekman Number $Ex = \frac{\text{Friction term}}{\text{Coriolis term}} = \frac{\alpha_x \frac{u}{L^2}}{f_0 u} = \boxed{\frac{\alpha_x}{f_0 L^2} = Ex}$ Important if $Ex \approx 1$

Normally, for the interior of the ocean
and for large scale motions

$$Ro \approx 10^3 \text{ and } Ex \approx 10^{-3}$$

$$\boxed{\begin{aligned} \frac{\alpha_x}{f_0 L^2} &= Ex \\ \frac{\alpha_y}{f_0 L^2} &= E_y \\ \frac{\alpha_z}{f_0 H^2} &= E_z \end{aligned}}$$

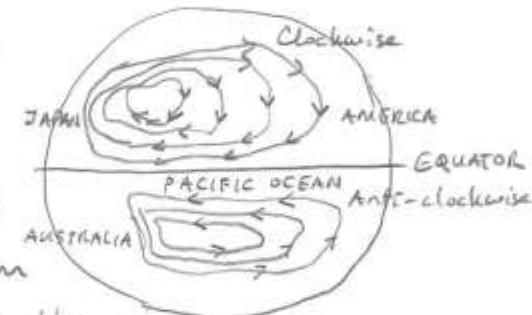
CURRENTS WITH FRICTION ; WIND DRIVEN CIRCULATION

A well-known feature of the surface-layer circulation is that it is clockwise in the northern hemisphere and anticlockwise in the southern hemisphere. The figure gives a very crude sketch of these circulation systems.

About 1875 Croall suggested that the frictional stress of the wind was the direct cause but he did not present any theory. In 1878, Zöppritz demonstrated quantitatively that the transfer of momentum and energy from wind to water was much too slow a process (on the order of months) to account for ocean currents. His demonstration was in error because he used the molecular coefficient of viscosity (i.e. friction) as determined in the laboratory for laminar (smooth) flow. In reality, the flow is almost invariably turbulent and in this type of flow the turbulent or "eddy" viscosity comes into play hence increasing the vertical transfer of momentum and energy to hundreds of thousands of times that due to molecular processes.

Making use of the eddy coefficient of viscosity concept, a series of developments followed:

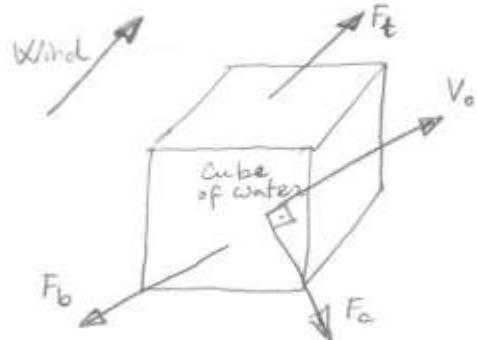
- 1) In 1898 Nansen explained qualitatively why wind-driven currents flow not in the direction of the wind but at 20° to 40° to the right of it (in the northern hemisphere).
- 2) In 1902 Ekman explained quantitatively for an idealized ocean how the rotation of the earth was responsible for the deflection of the current which Nansen had observed.
- 3) In 1947 Sverdrup showed how the main features of the equatorial surface currents could be attributed to the wind as a driving agent.
- 4) In 1948 Stommel explained the westward intensification of the wind-driven circulation.



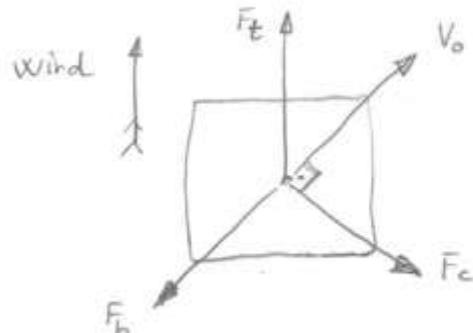
- 5) In 1950 Munk combined most of the above to obtain analytical expressions which described quantitatively the main features of the wind-driven circulation in terms of the real wind field
- 6) In recent years, numerous numerical models have been developed for the circulation of individual ocean areas and for the world ocean.

Nansen's qualitative argument

Nansen qualitatively explained why icebergs in the Arctic drifted in a direction to the right of the direction of the wind at the sea surface, not in the direction of the wind itself.



Perspective view



Plan view

F_t : Tangential force on the top surface of the cube of water due to wind

F_c : Coriolis force acting to the right of the motion

F_b : Retarding force of fluid friction on the bottom of the cube in a direction opposite to the motion

The combination of F_t and F_c would cause the cube to accelerate but as it does so the retarding force F_b increases. Finally, a steady-state is reached in which F_t , F_c and F_b are in balance. The cube moves at a steady speed V_o in some direction between F_t and F_c to the right of the wind direction.

The Equation of Motion with Friction Included

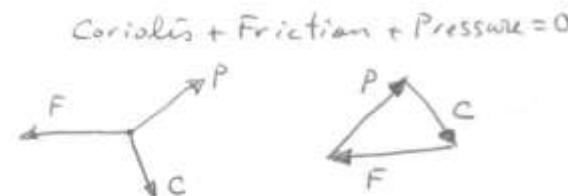
The horizontal equations of motion become, when friction is included (and the Coriolis term involving w is omitted)

$$\frac{du}{dt} = fv - \alpha \frac{\partial p}{\partial x} + F_x, \quad \frac{dv}{dt} = -fu - \alpha \frac{\partial p}{\partial y} + F_y$$

where F_x and F_y stand for the components of friction per unit mass in the fluid.

If there are no accelerations,

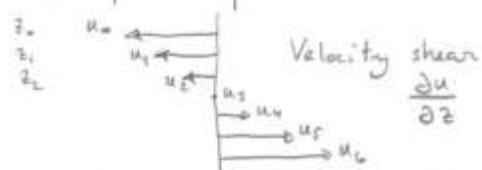
$$fv + F_x - \alpha \frac{\partial p}{\partial x} = 0, \quad -fu + F_y - \alpha \frac{\partial p}{\partial y} = 0$$



This situation differs from the geostrophic relationship in that, with the third force (friction) acting, the pressure and the Coriolis forces are no longer directly opposed.

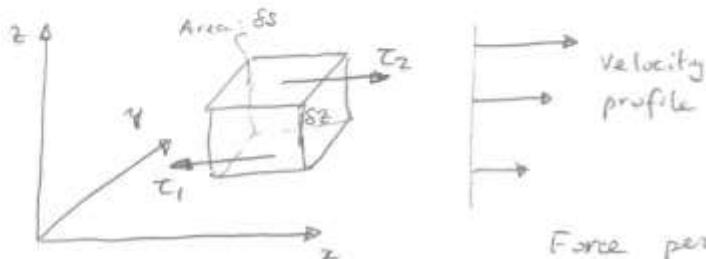
We must now write expressions for the frictional forces F_x and F_y . Newton's law of friction states that in a fluid, the friction stress τ , which is the force per unit area on a plane parallel to the flow, is given by

$$\tau = \mu \frac{du}{dz} = \rho v \frac{du}{dz}$$



The stress τ acts on the surface between the two layers which are moving at different speeds, tending to slow down the faster and to speed up the slower. The quantity μ is the coefficient of (molecular) dynamic viscosity, while $v = \mu/\rho$ is the coefficient of (molecular) kinematic viscosity. For sea water at 20°C $\mu = 10^{-3}$ kg/m.s, $v = 10^{-6}$ m²/s. These are the molecular values which apply to smooth laminar flow with $Re = \frac{UL}{v}$ less than 1000.

In the ocean, where the motion is generally turbulent, the effective value of the kinematic viscosity is the eddy viscosity $A_x \sim A_y \sim 10^5 \text{ m}^2/\text{s}$ for horizontal shear ($\partial u/\partial y, \partial v/\partial x$) or $A_z \sim 10^1 \text{ m}^2/\text{s}$ for vertical shear ($\partial u/\partial z$). $\tau = \rho A_z (\partial u/\partial z)$



$$\text{Force per unit volume} = \frac{(\tau_2 - \tau_1) \delta s}{\delta V} = \frac{\partial \tau}{\partial z} \quad \text{Force per unit mass} = \frac{1}{\rho} \frac{\partial \tau}{\partial z} = \alpha \frac{\partial \tau}{\partial z}$$

Force per unit mass $= \alpha \frac{\partial \tau}{\partial z} = \alpha \frac{\partial}{\partial z} \left(\rho A_z \frac{\partial u}{\partial z} \right)$. We use A_z because our derivation is for vertical shear ($\partial u/\partial z$ or $\partial v/\partial z$). Since we have little information regarding the variation of A_z with depth we assume A_z to be constant so that

$$\text{Friction force per unit mass} = \alpha A_z \frac{\partial^2 u}{\partial z^2}$$

The horizontal equations of motion become

$$fv + \alpha \frac{\partial \tau_x}{\partial z} = fv + A_z \frac{\partial^2 u}{\partial z^2} = \alpha \frac{\partial p}{\partial x}$$

$$-f u + \alpha \frac{\partial \tau_y}{\partial z} = -f u + A_z \frac{\partial^2 v}{\partial z^2} = \alpha \frac{\partial p}{\partial y}$$

The vertical equation reduces to the hydrostatic eqn., α does not appear in the above eqn., it is solved from the continuity eqn after u and v are determined.

Ekman's solution to the equation of motion with friction present

In the eqns just derived there are two causative forces for motion, the distribution of mass (density) which gives rise to the pressure terms, and the wind friction term. We can then think of the velocity as having two parts, one associated with the horizontal pressure gradient and one with vertical friction. Each part can be solved separately and the two added together.

$$fv = f(v_g + v_E) = \alpha \frac{\partial p}{\partial x} - A_z \frac{\partial^2}{\partial z^2} (u_g + u_E)$$

where $f v_g = \alpha \frac{\partial p}{\partial x}$, u_g, v_g being the geostrophic velocity components,

and $f v_E = -A_z \frac{\partial^2 u_E}{\partial z^2}$, u_E, v_E being the Ekman velocity components associated with vertical shear friction (non-geostrophic).

The term $-A_z \frac{\partial^2 u_g}{\partial z^2}$ is neglected because it is $\lesssim 10^3 \alpha (\partial p / \partial x)$ as shown in dimensional analysis.

To simplify the problem, Ekman (1905) assumed the water to be homogeneous and that there was no slope at the surface, so that the pressure terms would be zero and v_g therefore also zero, he solved for v_E only. At Nansen's suggestion, Ekman first studied the effect of the frictional stress at the sea surface due to the wind blowing over it. He assumed:

- 1) no boundaries; 2) infinitely deep water (to avoid the bottom friction terms);
- 3) A_z constant; 4) a steady wind blowing for a long time; 5) homogeneous water and the sea surface level so that $\partial p / \partial x = \partial p / \partial y = 0$; 6) f to be constant.

The equations then became

$$f v_E + A_2 \frac{\partial^2 u_E}{\partial z^2} = 0, \quad -f u_E + A_2 \frac{\partial^2 v_E}{\partial z^2} = 0$$

EKMAN EQUATIONS

Coriolis + Friction = 0

If, for simplicity, we assume the wind to be blowing in the y -direction, the solutions to Ekman's equations are

$$u_E = \pm V_o \cos \left(\frac{\pi}{4} + \frac{\pi}{D_E} z \right) \exp \left(\frac{\pi}{D_E} z \right) \quad (+ \text{ for northern hemisphere})$$

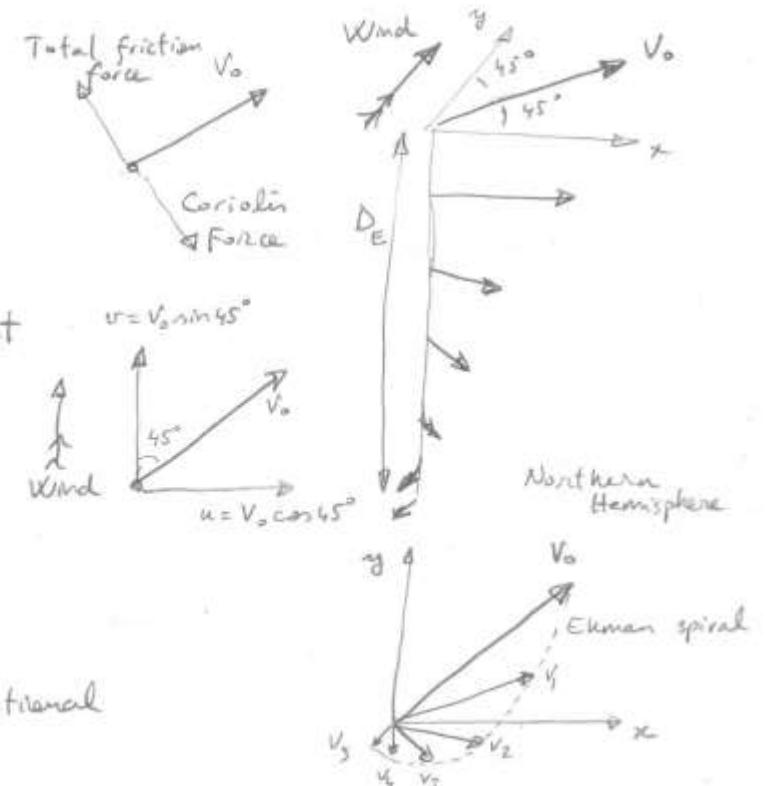
$$v_E = V_o \sin \left(\frac{\pi}{4} + \frac{\pi}{D_E} z \right) \exp \left(\frac{\pi}{D_E} z \right) \quad (- \text{ for southern hemisphere})$$

where $V_o = \sqrt{2} \pi \frac{T_{yr}}{D_E \rho |f|}$ is the total Ekman surface current

T_{yr} : magnitude of the wind stress on the sea surface
(approximately proportional to the wind speed squared
and acting in the direction of the wind)

$|f|$: the magnitude of f ,

$D_E = \pi \sqrt{2 A_2 / |f|}$ is the Ekman depth or depth of frictional influence



The solutions may be interpreted as follows

- 1) At the sea surface where $z=0$, the solutions become $u = \pm V_0 \cos 45^\circ$, $v = V_0 \sin 45^\circ$, which means that the surface current flows at 45° to the right (left) of the wind direction.
- 2) Below the surface, where z is no longer zero, the total current speed $= V_0 \exp(\bar{\pi}z/D_E)$ becomes smaller as depth increases, as z becomes more negative, while the direction changes clockwise (anticlockwise) in the northern (southern) hemisphere.
- 3) The direction of the flow becomes opposite to that at the surface at $z = -D_E$ where the speed has fallen to $\exp(-\bar{\pi}) = 0.04$ of that at the surface. The depth D_E is usually arbitrarily taken as the effective depth of the wind-driven current, the Ekman layer.

In order to obtain numerical relations between the surface current V_0 , the wind speed W , and the depth D_E , Ekman made use of two experimental observations:

Obs. 1 : The wind stress magnitude $\tau_y = \rho_a C_D W^2$ where ρ_a = the density of air, the drag coefficient $C_D \approx 1.4 \times 10^{-3}$ (non-dimensional), and W is the wind speed in m/s. Then,

$$\tau_y = 1.3 \text{ kg/m}^2 \times 1.4 \times 10^{-3} \times W^2 = 1.8 \times 10^{-3} W^2 \text{ (Pa)}$$

If we substitute this expression in $V_0 = \frac{\sqrt{2\bar{\pi}} \tau_y}{(D_E \rho_a f)}$

$$V_0 = \frac{\sqrt{2\bar{\pi}} \times 1.8 \times 10^{-3} W^2}{D_E \times 1025 \times 1 \text{ ft}} = 0.79 \times 10^{-5} \frac{W^2}{D_E f} \text{ (m/s)}$$

Obs. 2 Field observations analysed by Ekman indicate that the surface current and the wind speed are related as $\frac{V_0}{W} = \frac{0.0127}{\sqrt{\sin(\phi)}}$ outside $\pm 10^\circ$ latitude from the equator. Substituting this expression in the above eqn and using $f = 2\pi L \sin \phi$ we get $D_E = \frac{4.3 W}{\sqrt{\sin(\phi)}}$ (m) (with $W \text{ in } \frac{\text{m}}{\text{s}}$)

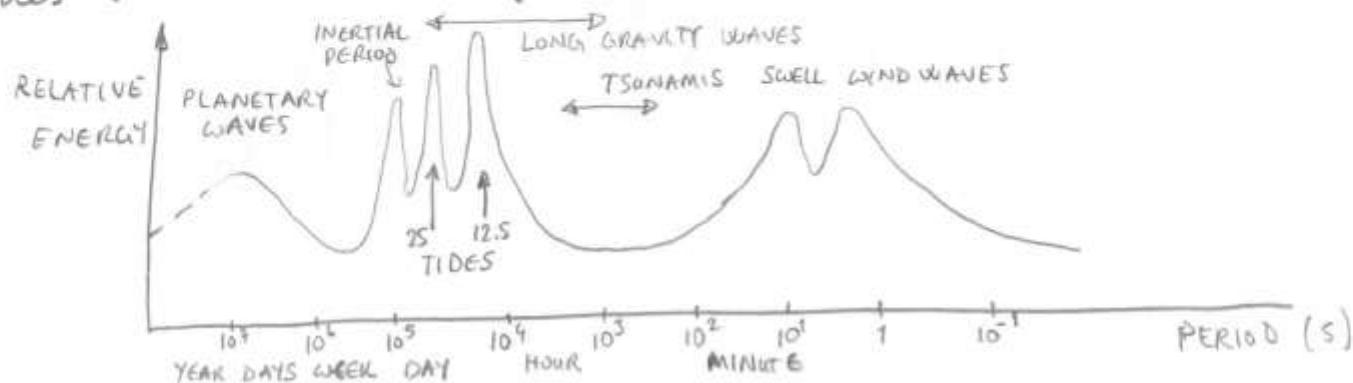
Therefore, if we know W at latitude ϕ we can calculate V_0 and D_E , and the velocity at any depth below D_E depends on W therefore A_z increases as W increases; if we know D_E we can estimate A_z .

$W = 10 \text{ m/s}$	$D_E = 100 \text{ m}$	50 m	45 m	$A_z = 0.014 \text{ m}^2/\text{s}$
$W = 20 \text{ m/s}$	$\phi = 10^\circ$	45°	30°	$A_z = 0.055 \text{ m}^2/\text{s}$
		$= 200 \text{ m}$	100 m	

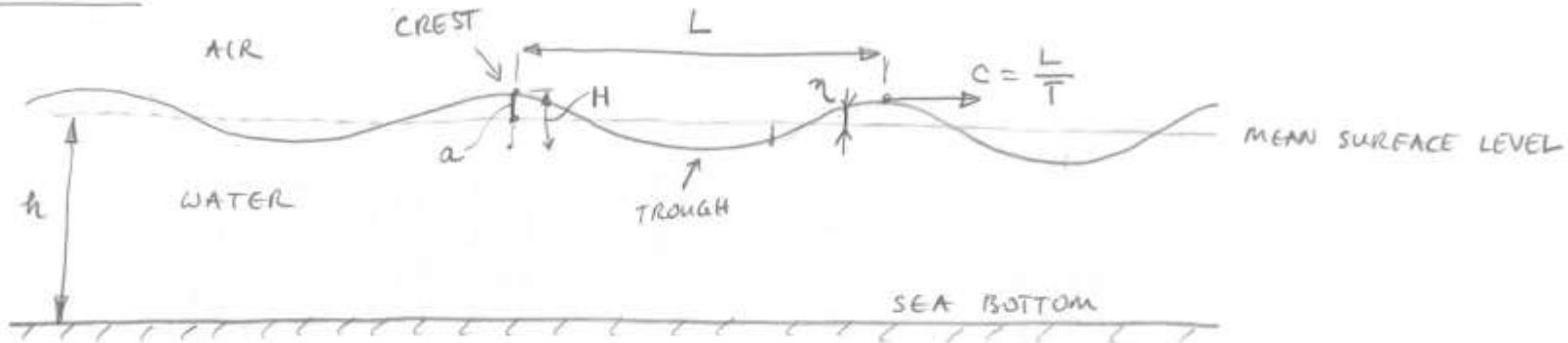
WAVES

The waves on the oceans may be classified under the following categories according to the generation mechanisms :

- 1) Ripples, wind waves, and swell : Due to the effect of the wind on the water surface
- 2) Internal waves : Due to the density variations coupled with currents.
- 3) Tsunamis : Due to seismic disturbances of the sea bottom or shore (landslide).
- 4) Gyroscopic - Gravity Waves (Surface and Internal) : Due to the Coriolis effect on long-period waves
- 5) Rossby Waves or Planetary Waves : Due to the effects of planetary gravity effects.
- 6) Tides : Due to fluctuating gravitational forces of the moon and the sun.



IDEAL WAVE PROFILE



L : Wavelength , T : Period , C: Celerity (speed) , H : Wave height = 2 a (amplitude)

h : Depth of water below mean surface level . $L = 1.56 T^2$

Period	Wavelength	Name
$L = 1.56 T^2$	Centimeters	Ripples
0 - 0.2 s	to about 130m	Wind Waves
0.2 - 9 s	hundreds of meters	Swell
9 s - 15 s	many hundreds of meters	Long swell or forerunners
15 s - 30 s	to thousands of kilometers	Long period waves including tsunamis
0.5 min - hours	thousands of kilometers	Tides
12.5 , 25 h, etc		

(Waves classified by periods)

BASICS OF WAVE THEORIES

It is always assumed that wave is irrotational :

$$\text{curl } \vec{V} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \vec{k} = 0$$

which implies $\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$, $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$

On the other hand, from the continuity eqn for constant ρ :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\vec{V} = \vec{\nabla} \cdot \Phi = \frac{\partial \Phi}{\partial x} \vec{i} + \frac{\partial \Phi}{\partial y} \vec{j} + \frac{\partial \Phi}{\partial z} \vec{k}$$

If we define a velocity potential Φ such that $\vec{V} = \vec{\nabla} \cdot \Phi$, we can easily show that the irrotationality conditions

or $u = \frac{\partial \Phi}{\partial x}$, $v = \frac{\partial \Phi}{\partial y}$, $w = \frac{\partial \Phi}{\partial z}$, we can easily show that the irrotationality conditions

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \Rightarrow \frac{\partial^2 \Phi}{\partial y \partial z} = \frac{\partial^2 \Phi}{\partial z \partial y}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \Rightarrow \frac{\partial^2 \Phi}{\partial z \partial x} = \frac{\partial^2 \Phi}{\partial x \partial z}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \Rightarrow \frac{\partial^2 \Phi}{\partial x \partial y} = \frac{\partial^2 \Phi}{\partial y \partial x}$$

$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$ $\Rightarrow \frac{\partial^2 \Phi}{\partial y \partial z} = \frac{\partial^2 \Phi}{\partial z \partial y}$, $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$ $\Rightarrow \frac{\partial^2 \Phi}{\partial z \partial x} = \frac{\partial^2 \Phi}{\partial x \partial z}$, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ $\Rightarrow \frac{\partial^2 \Phi}{\partial x \partial y} = \frac{\partial^2 \Phi}{\partial y \partial x}$ into the continuity eqn gives

are satisfied. Substituting $u = \frac{\partial \Phi}{\partial x}$, $v = \frac{\partial \Phi}{\partial y}$, $w = \frac{\partial \Phi}{\partial z}$ into the continuity eqn gives

the Laplace Equation : $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$

If we apply the approach of "separation of variables" to the Laplace eqn in 2-D:

$$\Phi = X(x) \cdot Z(z) \cdot T(t) \quad \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \Rightarrow X'' Z T + X Z'' T = 0 \quad \text{After multiplying the eqn by } \frac{1}{XZ} \text{ we get}$$

$$\frac{X''}{X} + \frac{Z''}{Z} = 0$$

In order to satisfy the above eqn we must have X''/X and Z''/Z constants with opposite signs. From physical considerations $\frac{Z''}{Z} = k^2$ and $\frac{X''}{X} = -k^2$

$$Z'' = k^2 Z, \quad D^2 = k^2, \quad D = \pm k, \quad Z = A e^{kz} + B e^{-kz}$$

$$X = \frac{\partial \Phi}{\partial z} = 0 \quad \text{at } z = -h; \quad X \cdot Z' T = 0 \quad \text{at } z = -h; \quad Z' = 0 \quad \text{at } z = -h$$

$$Z' = k A e^{kz} - k B e^{-kz} \quad \text{at } z = -h, \quad k A e^{-kh} - k B e^{kh} = 0, \quad B = A e^{-2kh}$$

$$Z = A [e^{kz} + e^{-kz-2kh}] = A e^{-kh} [e^{kz+kh} + e^{-kz-kh}] = \underset{A_0}{2A e^{-kh}} \left[\frac{e^{k(z+h)} + e^{-k(z+h)}}{2} \right]$$

$$Z = A_0 \operatorname{ch} k(z+h) \quad \left\{ Z'' = k^2 A_0 \operatorname{ch} k(z+h), \quad Z''/z = k^2; \quad Z' = k A_0 \operatorname{sh} k(z+h) = 0 \text{ at } z = -h \right\}$$

$Z(0) = A_0 \operatorname{ch} kh$. To normalize $Z(z) = A_0 \operatorname{ch} k(h+z)$ at $z = 0$ we must have

$$Z(z) = \frac{A_0 \operatorname{ch} k(h+z)}{A_0 \operatorname{ch} kh} = \frac{\operatorname{ch} k(h+z)}{\operatorname{ch} kh}$$

$$x'' = -k^2 X \quad , \quad D^2 = -k^2 \quad , \quad D = \pm ik \quad , \quad X = A e^{ikx} + B e^{-ikx}$$

$$e^{ikx} = \cos kx + i \sin kx \quad , \quad e^{-ikx} = \cos kx - i \sin kx \quad , \quad X = A(\cos kx + i \sin kx) + B(\cos kx - i \sin kx)$$

$$X = (A+B) \cos kx + i(A-B) \sin kx \quad . \quad \text{Let } A = a+ib \quad , \quad B = a-ib$$

$$X = 2a \cos kx + i(2ib) \sin kx = 2a \cos kx - 2b \sin kx = A^* \cos kx + B^* \sin kx$$

$$X = A_0 \cos(kx + \phi)$$

At this point we have $\Phi = T(t) \cdot A_0 \cos(kx + \phi) \cdot \frac{\operatorname{ch} k(h+z)}{\operatorname{ch} kh}$

We assume time-periodic waves with frequency " $\omega = \frac{2\pi}{T}$ " hence $T(t) = e^{i\omega t}$ which can be placed in the cosine term as $\cos(kx - \omega t + \phi)$ for right-going waves and as $\cos(kx + \omega t + \phi)$ for left-going waves. Considering the right-going waves only,

$$\Phi = A_0 \frac{\operatorname{ch} k(h+z)}{\operatorname{ch} kh} \cos(kx - \omega t + \phi)$$

Using the kinematic b.c. at $z=0$ (instead of η) $\frac{\partial \gamma}{\partial t} = w = \frac{\partial \phi}{\partial z}$

$$\frac{\partial \gamma}{\partial t} = A_0 \frac{k \operatorname{sh} k(h+0)}{\operatorname{ch} kh} \cos(kx - \omega t + \phi) = k A_0 \operatorname{th} kh \cos(kx - \omega t + \phi)$$

$$\eta = -\frac{k A_0}{\omega} (\operatorname{th} kh) \sin(kx - \omega t + \phi)$$

We must also satisfy the dynamic free surface condition $\rho(\eta) = 0$, ($z=0$)

$$\frac{\partial \phi}{\partial t} + g\eta = 0 + \omega A_0 \frac{\text{ch } k(h+z)}{\text{ch } kh} \sin(kx - \omega t + \phi) - g \frac{kA_0}{\omega} (\text{th } kh) \sin(kx - \omega t + \phi) = 0$$

$$\omega - \frac{gk}{\omega} \text{th } kh = 0 \Rightarrow \boxed{\omega^2 = gk \text{th } kh}$$

Linear Dispersion Relation.

Going back to $\eta = -\frac{kA_0}{\omega} \text{th } kh \sin(kx - \omega t + \phi) = -\frac{kA_0}{\omega} \frac{\omega z}{gk} \sin(kx - \omega t + \phi)$

$$\eta = \underbrace{-\frac{\omega}{g} A_0}_{a_0} \sin(kx - \omega t + \phi) = \underline{a_0 \sin(kx - \omega t + \phi)}$$

$A_0: \frac{m^2}{s}, \quad \omega: \frac{rad}{s}, \quad g: \frac{m}{s^2}$
 $a_0: m$

$$a_0 = -\frac{\omega}{g} A_0, \quad A_0 = -\frac{g}{\omega} a_0, \quad \boxed{\phi = -\frac{g}{\omega} a_0 \frac{\text{ch } k(h+z)}{\text{ch } kh} \cos(kx - \omega t + \phi)}$$

$$u = \frac{\partial \phi}{\partial x} = \frac{gk}{\omega} a_0 \frac{\text{ch } k(h+z)}{\text{ch } kh} \sin(kx - \omega t + \phi) = \frac{gk}{\omega} \frac{\text{ch } k(h+z)}{\text{ch } kh} \cdot \eta$$

$$w = \frac{\partial \phi}{\partial z} = -\frac{gk}{\omega} a_0 \frac{\text{sh } k(h+z)}{\text{ch } kh} \cos(kx - \omega t + \phi)$$

$$\rho = \frac{\partial \phi}{\partial t} = \rho g a_0 \frac{\text{ch } k(h+z)}{\text{ch } kh} \sin(kx - \omega t + \phi) = \rho g \frac{\text{ch } k(h+z)}{\text{ch } kh} \eta$$

ORBITAL CHARACTERISTICS OF VELOCITIES

$$\omega^2 = gk \frac{sh\kh}{ch\kh} , \quad \frac{\omega}{sh\kh} = \frac{gk}{\omega} \cdot \frac{1}{ch\kh}$$

Using this relation we can re-write

u and ω as

$$u = wa_0 \frac{ch\kh(h+z)}{sh\kh} \sin(kx - \omega t + \phi) , \quad \omega = -wa_0 \frac{sh\kh(h+z)}{sh\kh} \cos(kx - \omega t + \phi)$$

$$\left(\frac{u}{wa_0}\right)^2 / \left[\frac{ch\kh(h+z)}{sh\kh}\right]^2 = \sin^2(kx - \omega t + \phi) , \quad \left(\frac{\omega}{wa_0}\right)^2 / \left[\frac{sh\kh(h+z)}{sh\kh}\right]^2 = \cos^2(kx - \omega t + \phi)$$

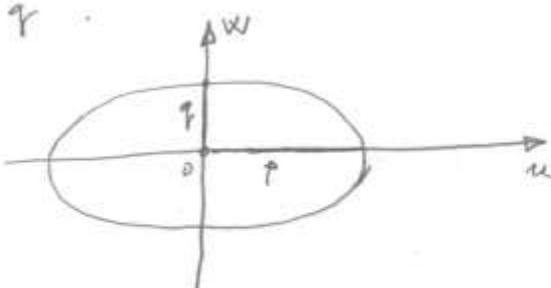
Adding the two eqns side by side and using the relation $\sin^2(kx - \omega t + \phi) + \cos^2(kx - \omega t + \phi) = 1$

gives

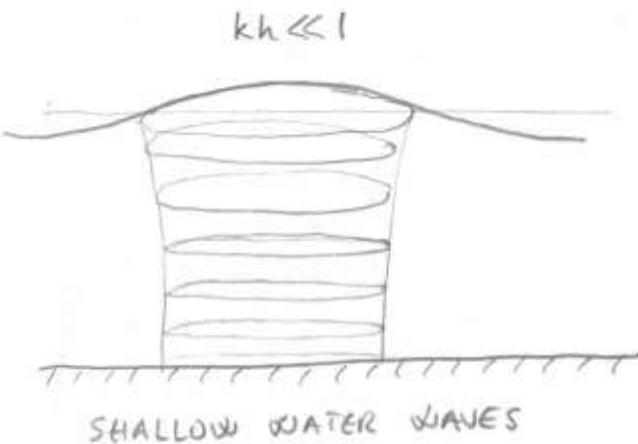
$$\frac{u^2}{\left[wa_0 \frac{ch\kh(h+z)}{sh\kh}\right]^2} + \frac{\omega^2}{\left[wa_0 \frac{sh\kh(h+z)}{sh\kh}\right]^2} = 1$$

Let $p \triangleq wa_0 \frac{ch\kh(h+z)}{sh\kh}$, $q \triangleq wa_0 \frac{sh\kh(h+z)}{sh\kh}$ then

$u^2/p^2 + \omega^2/q^2 = 1$ which is the equation of an ellipse with axes p and q .



Since p and q are functions of "z" and "kh" we can examine various cases corresponding to special limits of "kh".



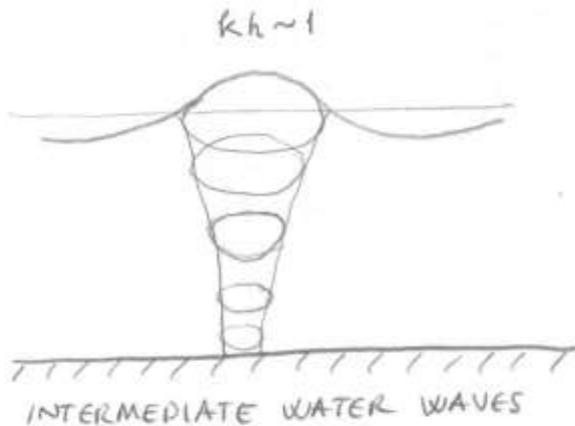
SHALLOW WATER WAVES

$$kh \ll 1 \Rightarrow Sh \ll kh \sim kh$$

$$kh \ll 1 \Rightarrow Ch \ll kh \sim 1$$

$$p \rightarrow w_a, q \rightarrow w_a, \frac{1}{kh}$$

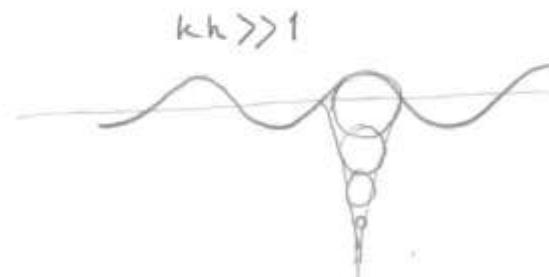
$$\frac{p}{q} \rightarrow \frac{1}{kh} \gg 1$$



INTERMEDIATE WATER WAVES

$$p = w_a, \frac{Ch k (h+z)}{Sh kh}$$

$$q = w_a, \frac{Sh k (h+z)}{Sh kh}$$



DEEP WATER WAVES

$$kh \gg 1 \Rightarrow Sh \ll kh \rightarrow +\infty$$

$$kh \gg 1 \Rightarrow Ch \ll kh \rightarrow +\infty$$

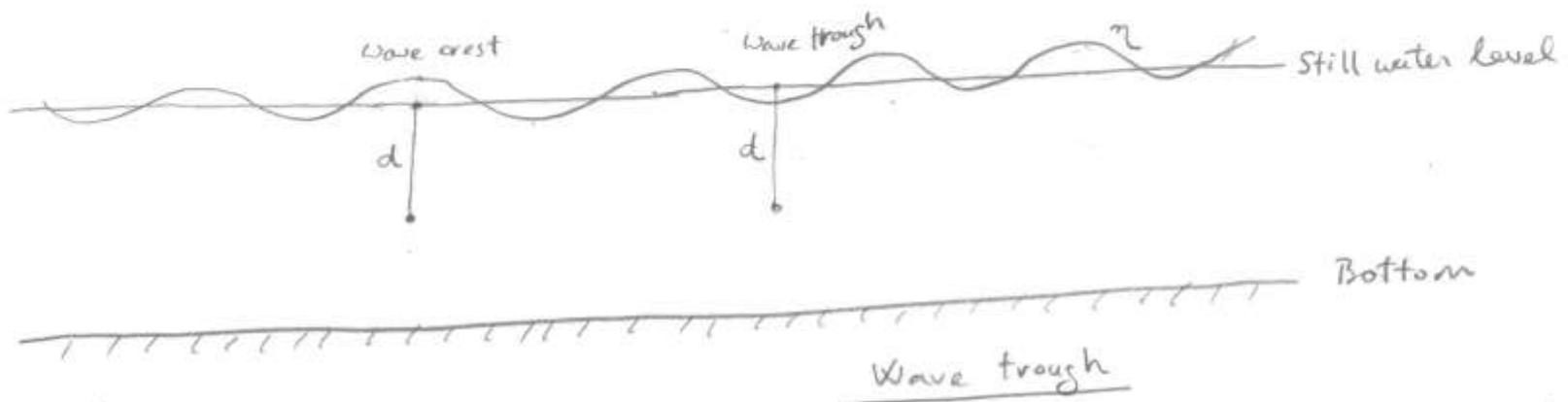
$$p \rightarrow w_a, q \rightarrow w_a$$

$$\frac{p}{q} \rightarrow 1 \text{ (Circle)}$$

COMPARISON BETWEEN THE DYNAMIC WAVE PRESSURE AND ITS HYDROSTATIC EQUIVALENT

The dynamic wave pressure according to linear theory is

$$\frac{p}{\rho} = g a_0 \frac{ch k (h+z)}{ch kh} \sin(kx - \omega t + \phi) = g \frac{ch k (h+z)}{ch kh} n$$



Wave Crest

Total pressure : Hydrostatic + Hydrodynamic

$$p_D = \rho g d + \rho g a_0 \frac{ch k (h-d)}{ch kh}$$

Total hydrostatic pressure

$$p_S = \rho g d + \rho g a_0$$

$$\text{Since } \frac{ch k (h-d)}{ch kh} < 1 \Rightarrow p_D < p_S$$

Wave trough

Total pressure : Hydrostatic + Hydrodynamic

$$p_D = \rho g d - \rho g a_0 \frac{ch k (h-d)}{ch kh}$$

Total hydrostatic pressure

$$p_S = \rho g d - \rho g a_0$$

$$\text{Since } \frac{ch k (h-d)}{ch kh} < 1 \Rightarrow \underline{\underline{p_D > p_S}}$$

SPECIAL FORMS OF THE DISPERSION RELATION

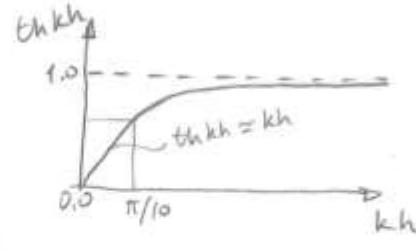
In the most general form we have $\omega^2 = gk \tanh kh$ for waves on arbitrary depths. In some special cases it is advantageous to use the simplified form of this relation, we shall now consider these special cases.

i) Shallow water waves If $\frac{h}{\lambda} < \frac{1}{20}$ or $kh < \frac{\pi}{10}$ then the wave is called

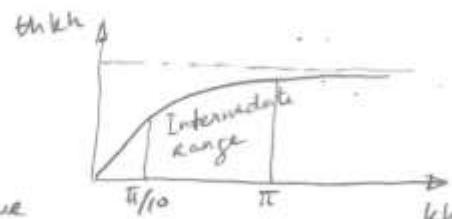
a shallow water wave. For small arguments $\tanh kh \approx kh$

$$\omega^2 = gk \tanh kh \Rightarrow \omega^2 = gk \cdot kh , \quad \underline{\omega^2 = gk^2 h}$$

$$\frac{\omega^2}{k^2} = gh , \quad \frac{k^2 c^2}{k^2} = gh , \quad c^2 = gh , \quad \boxed{c = \sqrt{gh}}$$



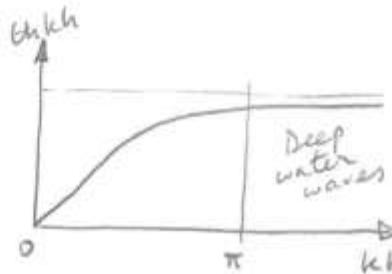
ii) Intermediate water waves: If $\frac{1}{20} < \frac{h}{\lambda} < \frac{1}{2}$ or $\frac{\pi}{10} < kh < \pi$ then the wave is called an intermediate water wave and $\omega^2 = gk \tanh kh$, $c^2 = \frac{g}{k} \tanh kh$



iii) Deep water waves: If $\frac{h}{\lambda} > \frac{1}{2}$ or $kh > \pi$ then the wave is called a deep water wave. For large arguments $\tanh kh \rightarrow 1$

$$\omega^2 = gk \tanh kh \Rightarrow \omega^2 = gk \cdot 1 , \quad \underline{\omega^2 = gk}$$

$$c^2 = \frac{g}{k} , \quad c = \sqrt{\frac{g}{k}} , \quad \boxed{c = \sqrt{\frac{g\lambda}{2\pi}}}$$



WAVE GROUPS & GROUP VELOCITY

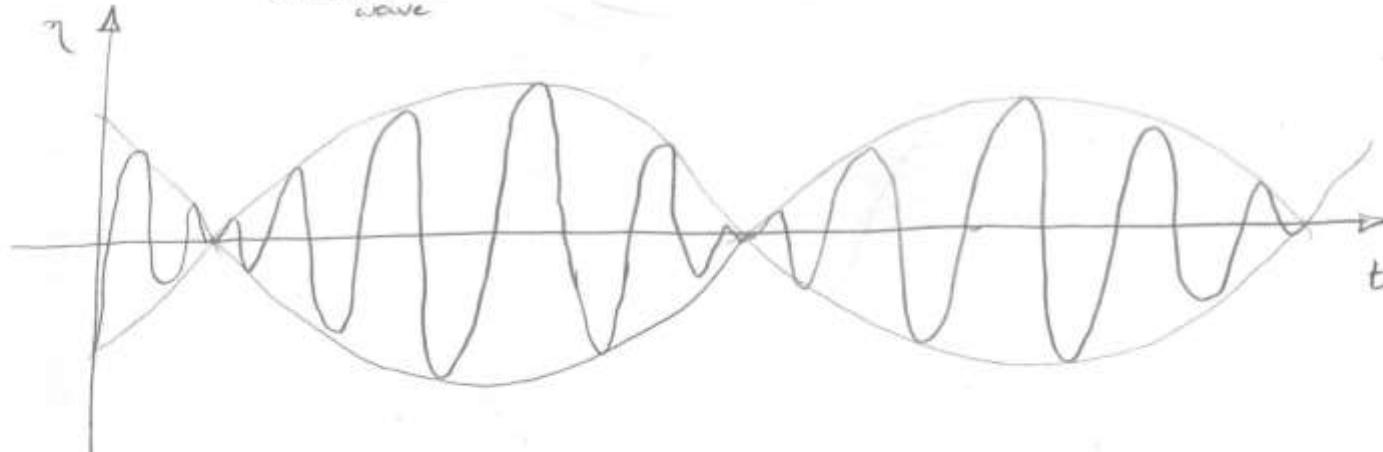
Since the real waves always travel as groups it is necessary to examine their behaviour as groups. We can construct a simple group of waves by simply superposing two wave trains with slightly different wave numbers and frequencies.

$$\eta_1 = \cos(k_1 x - \omega_1 t), \quad \eta_2 = \cos(k_2 x - \omega_2 t)$$

where, for simplicity $a_1 = a_2 = 1$, and $k_1 = k + \Delta k$, $\omega_1 = \omega + \Delta\omega$, $k_2 = k - \Delta k$, $\omega_2 = \omega - \Delta\omega$

$$\eta = \eta_1 + \eta_2 = \cos[(k + \Delta k)x - (\omega + \Delta\omega)t] + \cos[(k - \Delta k)x - (\omega - \Delta\omega)t]$$

$$\eta = 2 \underbrace{\cos(kx - \omega t)}_{\text{Individual wave}} \underbrace{\cos(\Delta k x - \Delta\omega t)}_{\text{Envelope}}$$



Since we defined the velocity as $c = \frac{\omega}{k}$, similarly for the group velocity (velocity of the wave envelope) $c_g = \frac{\Delta\omega}{\Delta k}$.

For the limiting case of $\Delta k \rightarrow \Delta\omega \rightarrow 0$ we can define the group velocity as

$$c_g = \frac{d\omega}{dk} \quad \text{where} \quad \omega = \sqrt{gk \tanh kh}$$

$$c_g = \frac{c}{2} \left(1 + \frac{2kh}{\sinh 2kh} \right) \quad \text{with} \quad c = \sqrt{\frac{g}{k} \tanh kh}$$

SPECIAL FORMS OF THE GROUP VELOCITY

i) Shallow water waves ($kh < \pi/10$) $\sinh 2kh \rightarrow 2kh$ and $c = \sqrt{gh}$

$$c_g = c = \sqrt{gh}$$

ii) Intermediate water waves ($\pi/10 < kh < \pi$)

$$c_g = \frac{c}{2} \left(1 + \frac{2kh}{\sinh 2kh} \right) \quad \text{and} \quad c = \sqrt{\frac{g}{k} \tanh kh}$$

iii) Deep water waves ($kh > \pi$) $\sinh 2kh \rightarrow +\infty$ as $kh \gg 1$, $\frac{2kh}{\sinh 2kh} \rightarrow 0$

$$c = \sqrt{\frac{g}{k}},$$

$$c_g = \frac{i}{2} c = \frac{i}{2} \sqrt{\frac{g}{k}} = \frac{i}{2} \sqrt{\frac{g\lambda}{2\pi}}$$

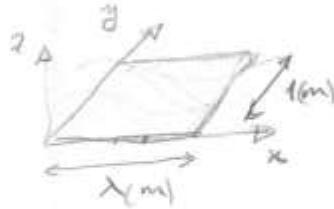
MEAN WAVE ENERGY

$$K.E. = \frac{1}{T} \int_0^T \left[\frac{1}{2} \rho \int_{-h}^h (u^2 + \omega^2) dz \right] dt, \quad P.E. = \frac{1}{T} \int_0^T \left[\int_0^h \rho g z dz \right] dt$$

using the relations for u , ω , and η as given by the linear theory we get

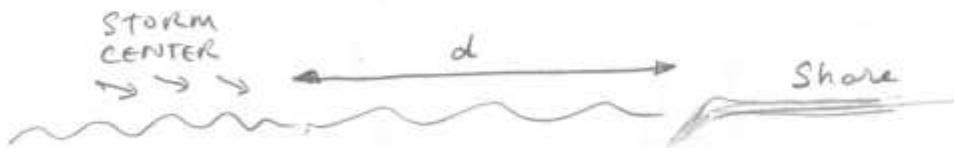
$$K.E. = \frac{1}{4} \rho g a^2, \quad P.E. = \frac{1}{4} \rho g a^2 \quad \text{where } a \text{ is the wave amplitude}$$

$$\text{Total mean energy is then } T.E. = K.E. + P.E. = \frac{1}{2} \rho g a^2 = \frac{1}{8} \rho g H^2 \left(\frac{\text{Joule}}{\text{m}^2} \right)$$



If we consider an area of λ (m) by 1 (m), respectively in the x - and y -directions. Then, the total mean wave energy in this area is

$$T.E. = \frac{1}{2} \rho g a^2 \underbrace{\lambda}_{\text{m}} \underbrace{1}_{\text{m}} \quad (\text{Joule})$$

PROBLEM

On a shore the forerunners of a distant storm with mean period of $T=12\text{s}$ is observed first. After 3 hours, a new group of waves with $T=8\text{s}$ arrives. Supposing that the two groups of waves were generated by the same storm in the open sea, compute the distance of the storm center from the shore and the time storm broke.

Since no water depth is given we make the plausible assumption that all those waves are deep water waves. Accordingly, $\omega^2 = gk$.

$$\text{Using } \omega = k \cdot c, \quad \omega \cdot \omega = gk, \quad \omega \cdot kc = gk \quad c = \frac{g}{\omega} = \frac{g}{\frac{2\pi}{T}} = \frac{gT}{2\pi}$$

$$c = \frac{gT}{2\pi} \text{ for deep water waves.}$$

$$\text{The first group of waves had } T = 12 \text{ s period, then } C_{12} = \frac{gT}{2\pi} = \frac{9.81 \times 12}{2\pi} = 18.74 \text{ m/s}$$

$$\text{In deep water } C_{g12} = \frac{1}{2} C_{12} = \frac{1}{2} \times 18.74 = 9.37 \text{ m/s}$$

$$\text{The second group of waves had } T = 8 \text{ s period, then } C_8 = \frac{gT}{2\pi} = \frac{9.81 \times 8}{2\pi} = 12.50 \text{ m/s}$$

$$\text{In deep water } C_{g8} = \frac{1}{2} C_8 = \frac{1}{2} \times 12.50 = 6.25 \text{ m/s}$$

Since the waves originate from the same storm they travel the same distance d .

$$d = C_{g12} \times t, \quad d = C_{g8} \times (t+3)$$

$$\text{Taking the ratio } \frac{C_{g12} t}{C_{g8}(t+3)} \text{ we get } \frac{C_{g12} t}{C_{g8}(t+3)} = 1, \quad \left(\frac{C_{g12}}{C_{g8}} \right) t = t+3$$

$$\left[\left(\frac{C_{g12}}{C_{g8}} - 1 \right) \right] t = 3 \text{ hours}, \quad t = \frac{3}{\left(\frac{C_{g12}}{C_{g8}} - 1 \right)} = \frac{3}{\left(\frac{9.37}{6.25} - 1 \right)}, \quad t = 6 \text{ hours}$$

$$d = C_{g12} \times t = 9.37 \frac{\text{m}}{5} \times 6 \times 60 \times 60 = 202392 \text{ m} \simeq \underline{\underline{200 \text{ km}}}$$

The distance of the storm from the shore.

PROBLEM

A satellite determines the wavelength of deep water waves as 312 m. When these waves approach to the shore they shorten to 200 m. What is the water depth at this point?



$$\text{For deep water waves } \omega^2 = gk_0, \left(\frac{2\pi}{T}\right)^2 = g \left(\frac{2\pi}{\lambda_0}\right), T = \sqrt{\frac{2\pi \lambda_0}{g}} \quad (\text{Deep water})$$

$$T = \sqrt{\frac{2\pi \times 312}{9.81}} = 14.1 \text{ sec. The period remains the same for all depths.}$$

$$\text{For an arbitrary water depth "h" we have } \omega^2 = gk_0 th kh. \text{ Using this relation}$$

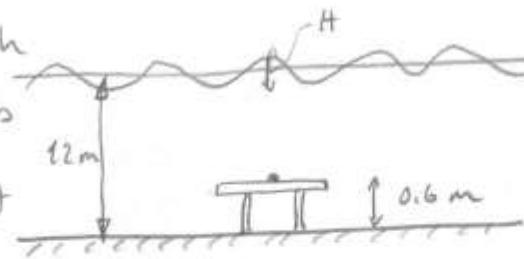
$$\left(\frac{2\pi}{14.1}\right)^2 = g \frac{2\pi}{200} th \left(\frac{2\pi h}{200}\right) \Rightarrow 0.64432 = th \left(\frac{2\pi h}{200}\right), 0.7655 = \frac{2\pi h}{200}$$

$h = 24.4 \text{ m}$ Lets check if we made the assumption of shallow water,

$$\omega^2 = gk^2h, k = \frac{1}{\lambda} \left(\frac{\omega}{g}\right)^2, h = \frac{1}{g} \left(\frac{\lambda}{T}\right)^2 = \frac{1}{9.81} \left(\frac{200}{14.1}\right)^2 = 20.5 \text{ m, which is}$$

different from the true result 24.4m. The ratio $h/\lambda = 24.4/200 = 0.122$ is slightly greater than 0.1 (the shallow water ratio) therefore result is not quite correct.

PROBLEM: A pressure gage in water of 12 m depth is placed 0.6 m above the bottom. The gage records a maximum of 124 kPa pressure at the frequency $f = 0.0666 \text{ Hz}$. Compute the wave height on the water surface?



$$f = 0.0666 \text{ Hz} \Rightarrow T = \frac{1}{f} = \frac{1}{0.0666} = 15 \text{ s}, \quad \omega = \frac{2\pi}{T} = 2\pi f = 2\pi \times 0.0666 = 0.4188 \frac{\text{rad}}{\text{s}}$$

From $\omega^2 = gk \tanh(kh)$, $(0.4188)^2 = 9.81 \tanh(12k)$, $k = \frac{0.017879}{\tanh(12k)}$ After several iterations $k \approx 0.04 \text{ rad/m}$. (For shallow water we have $\omega^2 = gk^2 h$, $k = \frac{\omega}{\sqrt{gh}} = \frac{0.4188}{\sqrt{9.81 \times 12}}$ $k \approx 0.038$). From the expression given for total pressure under a wave,

$$P_{\text{tot}} = \gamma g \left[\underbrace{\frac{ch k(h+z)}{ch kh}}_{\substack{\text{Hydrodynamic} \\ \text{wave pressure} \\ + or - according \\ to the sign of } \eta} \eta - z \right]$$

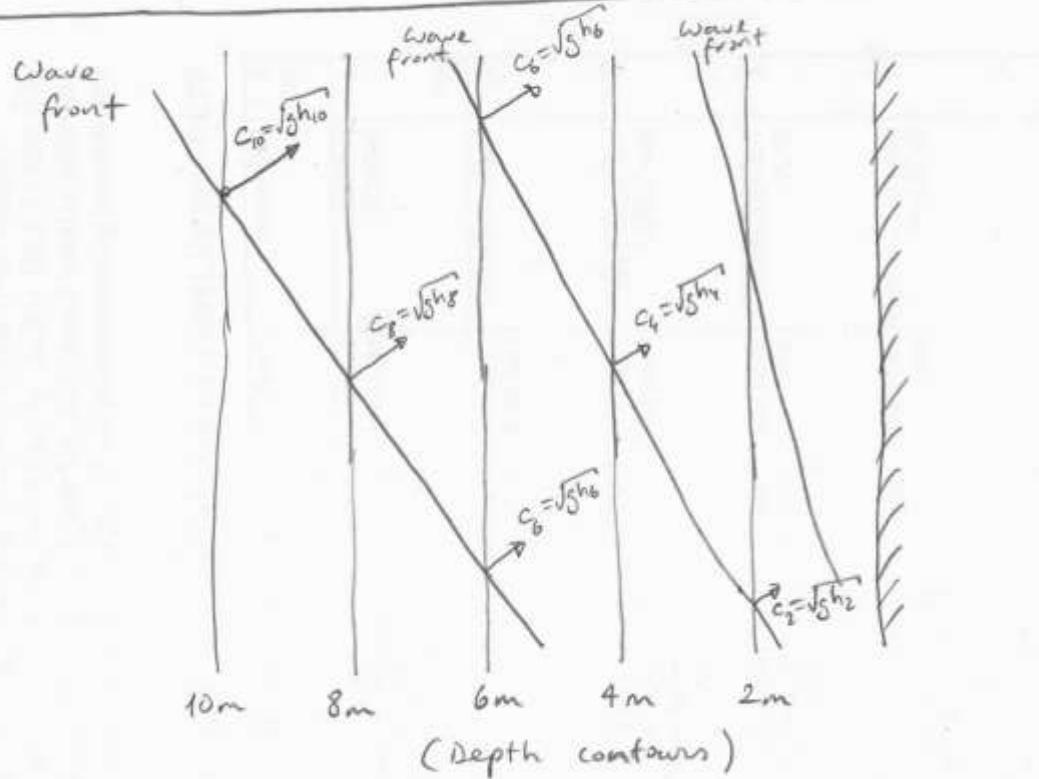
Hydrostatic
water pressure
Always + since z is always - below sea surface.

To make the pressure maximum, η must be maximum, which is $\eta = +H/2$ (wave crest).

$$124 \times 10^3 \text{ Pa} = 1025 \times 9.81 \times \left[\frac{ch 0.04(12 - H/2)}{ch(0.04 \times 12)} \cdot \frac{H}{2} - (-H/2) \right] \quad \text{Solving for } H \text{ gives}$$

$$H = 2.08 \text{ m}$$

REFRACTION OF WATER WAVES IN SHALLOW WATER



For simplicity we consider a nearshore region with depth contours parallel to the shoreline. A wave front moving obliquely towards the shore experiences different phase velocities at different parts of the front. The part of the wave front in deeper region moves faster since $c = \sqrt{gh}$ hence c is greater for greater h . These differences in wave celerity cause the wave front align itself with the shoreline. Such an effect is called "refraction".

CONSERVATION OF ENERGY FLUX WITHIN RAYS

Wavenumber vector indicates the direction of wave propagation and its magnitude is $2\pi/\lambda$ as defined before. $|\vec{k}| = 2\pi/\lambda$. The conservation of energy flux within rays states that the energy flux within any two rays is constant:

$$\dot{E} = E \cdot c_g \cdot b = \frac{1}{2} \rho g a^2 c_g \cdot b = \text{Constant}$$

Considering two locations as 1 and 2 we can write

$$[a^2 c_g b]_1 = [a^2 c_g b]_2 = \text{Constant}$$

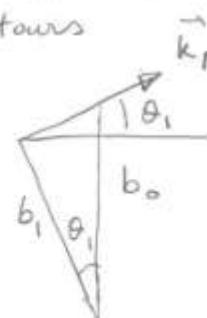
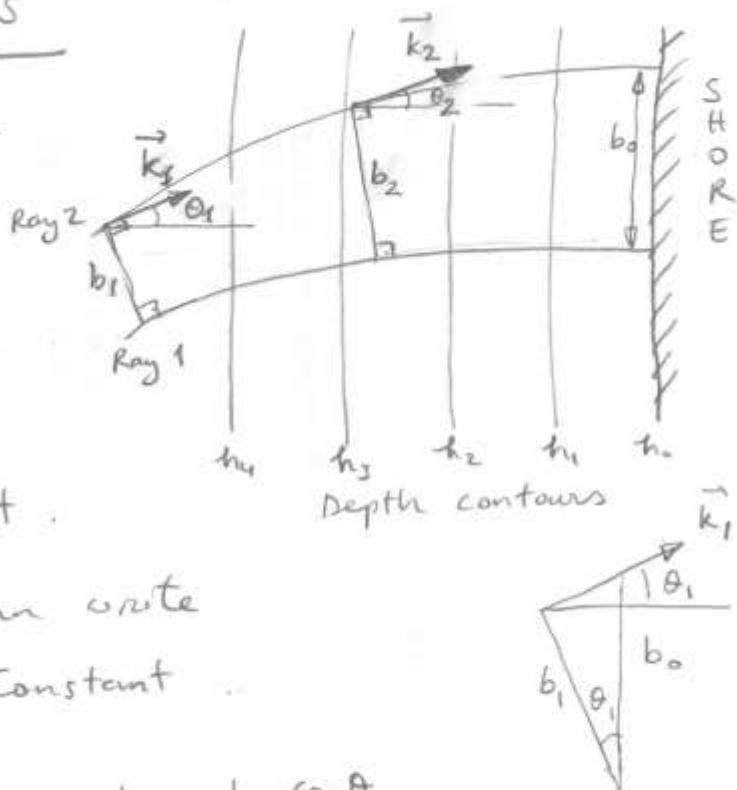
From the figure we can write $b_1 = b_0 \cos \theta_1$, $b_2 = b_0 \cos \theta_2$

$$a_1^2 c_{g1} b_0 / \cos \theta_1 = a_2^2 c_{g2} b_0 / \cos \theta_2$$

$$a_1^2 c_{g1} \cos \theta_1 = a_2^2 c_{g2} \cos \theta_2 \quad \text{since } c_g = \frac{1}{2} c \left(1 + \frac{2kh}{\sin 2kh} \right)$$

We define a new function $G = \frac{2kh}{\sin 2kh}$ so that $c_g = \frac{1}{2} c (1+G)$

$$a_1^2 (1+G_1) c_1 \cos \theta_1 = a_2^2 (1+G_2) c_2 \cos \theta_2$$



Let $c_1 = \omega/k_1$ and $c_2 = \omega/k_2$ (ω is constant since wave period T is constant).

$$a_1^2 (1+G_1) \frac{\omega}{k_1} \cos \theta_1 = a_2^2 (1+G_2) \frac{\omega}{k_2} \cos \theta_2$$

$$a_1^2 (1+G_1) \cos \theta_1 = a_2^2 \frac{k_1}{k_2} \cos \theta_2$$

From the dispersion relation $\omega^2 = g k \operatorname{th} k h$, again recalling that $\omega_1 = \omega_2 = \omega = \text{constant}$

$$g k_1 \operatorname{th} k_1 h_1 = g k_2 \operatorname{th} k_2 h_2 \quad \text{which gives}$$

$$\frac{k_1}{k_2} = \frac{\operatorname{th} k_2 h_2}{\operatorname{th} k_1 h_1}$$

Using this relation in the above eqn gives

$$a_1^2 (1+G_1) \cos \theta_1 = a_2^2 \frac{\operatorname{th} k_2 h_2}{\operatorname{th} k_1 h_1} \cos \theta_2$$

Solving for a_2 ,

$$a_2 = a_1 \left[\frac{\cos \theta_1 (1+G_1) \operatorname{th} k_1 h_1}{\cos \theta_2 (1+G_2) \operatorname{th} k_2 h_2} \right]^{1/2}$$

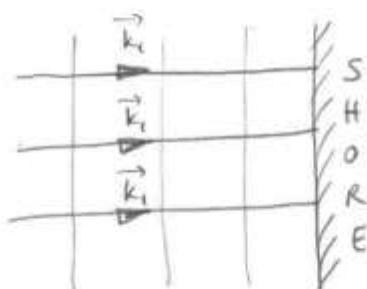
The above relation allows us to compute the wave amplitude at a point (2) using the given values at point (1). The effects of refraction due to varying depth contours are taken into account.

Let's assume that the computation starts from deep water. Point ② is now considered as an arbitrary point without any index. In deep water $k_i h_i \rightarrow \infty$ hence $\text{th } k_i h_i \rightarrow 1$, $a_i \rightarrow 0$. Denoting the deep water wave amplitude by a_0 instead of a_i ; likewise θ_i by θ_0 we get

$$a = a_0 \left[\frac{\cos \theta_0}{\cos \theta (1+a) \text{th } kh} \right]^{1/2}$$

giving the amplitude of a wave moving from deep water to intermediate and shallow depths. Here, θ_0 is the angle wavenumber vector makes with the horizontal axis.

If the wave incidence is normal to the depth contours and further if the depth contours are all parallel to the shoreline $\cos \theta_0 = \cos \theta = 1$ since $\theta_0 = \theta = 0^\circ$



$$a = a_0 [(1+a) \text{th } kh]^{-1/2}$$

where $a = 2kh / \text{sh } 2kh$.

From deep to intermediate or shallow waters.

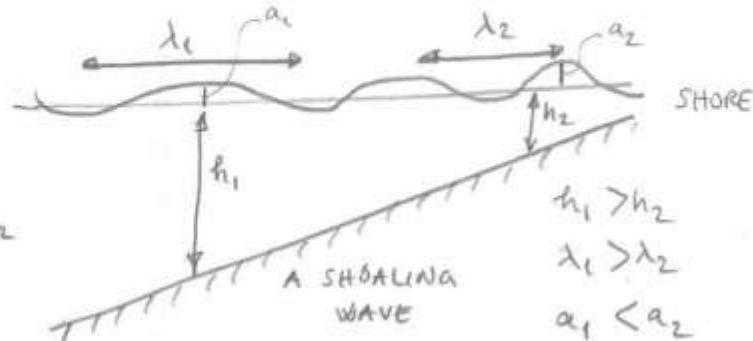
For waves moving from shallow water to shallow water (like Tsunamis) we can write, again for $\theta_1 = \theta_2 = 0$,

$$a_2 = a_1 \left[\frac{(1+G_1) \operatorname{th} k_1 h_1}{(1+G_2) \operatorname{th} k_2 h_2} \right]^{1/2}$$

In shallow waters $kh \ll 1$, $\operatorname{th} k_i h_i \approx k_i h_i$, $\operatorname{th} k_2 h_2 = k_2 h_2$

$$G_1 = G_2 \approx 1$$

$$a_2 = a_1 \left(\frac{k_1 h_1}{k_2 h_2} \right)^{1/2}$$



On the other hand, from the dispersion relation $\frac{k_1}{k_2} = \frac{\operatorname{th} k_2 h_2}{\operatorname{th} k_1 h_1}$. For shallow waters

$\frac{k_1}{k_2} = \frac{k_2 h_2}{k_1 h_1}$ or $\left(\frac{k_1}{k_2}\right)^2 = \frac{h_2}{h_1}$ (We could obtain the same relation from the dispersion relation for shallow waters : $\omega^2 = g k^2 h$. $\omega_1 = \omega_2 \Rightarrow g k_1^2 h_1 = g k_2^2 h_2$, $\left(\frac{k_1}{k_2}\right)^2 = \frac{h_2}{h_1}$)

$$a_2 = a_1 \left[\sqrt{\frac{h_2}{h_1}} \cdot \frac{h_1}{h_2} \right]^{1/2}, \quad \boxed{a_2 = a_1 \left(\frac{h_1}{h_2} \right)^{1/4}}$$

We could get the same relation from the constancy of energy flux in shallow water :

$$\frac{1}{2} \rho g a_1^2 \sqrt{g h_1} = \frac{1}{2} \rho g a_2^2 \sqrt{g h_2}, \quad a_2 = a_1 \left(\frac{h_1}{h_2} \right)^{1/4}.$$

An important relation for computing the refractive angle is Snell's law which states that

$k_1 \sin \theta_1 = \text{Constant}$
$k_1 \sin \theta_1 = k_2 \sin \theta_2$

As an example, consider a tsunami generated at $h_1 = 3000$ m depth which moves to $h_2 = 5$ m depth. Since tsunami is a shallow water wave everywhere

$$\frac{a_2}{a_1} = \left(\frac{h_1}{h_2}\right)^{1/4} = \left(\frac{3000}{5}\right)^{1/4} = 5$$

Thus, a tsunami with $a_1 = 1$ m amplitude reaches $a_2 = 5$ m amplitude at water depth $h_2 = 5$ m

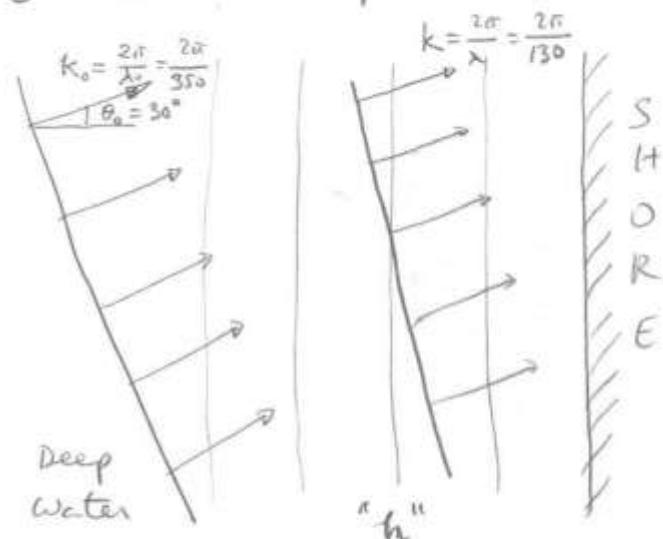
PROBLEM: A wave with amplitude $a_0 = 1$ m and wavelength $\lambda_0 = 350$ m moves into a region of parallel depth contours with incident wave angle $\theta_0 = 30^\circ$. At an arbitrary depth "h" the wavelength is determined to be $\lambda = 130$ m. Compute the wave period, water depth "h", the wave angle θ and wave amplitude a at depth "h".

Since we know the deep water wavelength, we can compute the wave period from the relation

$$T = \sqrt{\frac{2\pi \lambda_0}{g}} = \sqrt{\frac{2\pi \times 350}{9.81}} = 15.0 \text{ seconds}$$

Since the wave period is constant we can use again the dispersion relation to compute the water depth "h" where λ becomes 130 meters.

$$\omega^2 = gk \tanh kh, \quad \left(\frac{2\pi}{15}\right)^2 = 9.81 \times \frac{2\pi}{130} \tanh\left(\frac{2\pi h}{130}\right), \quad h = \frac{130}{2\pi} \tanh^{-1}(0.37) \Rightarrow h = 8 \text{ m}$$



From Snell's law $k \cdot \sin \theta = \text{constant}$, (56)

$$k_0 \sin \theta_0 = k \sin \theta$$

$$\frac{2\pi}{\lambda_0} \sin \theta_0 = \frac{2\pi}{\lambda} \sin \theta$$

$$\frac{2\pi}{350} \sin 30^\circ = \frac{2\pi}{130} \sin \theta$$

$$0.1857 = \sin \theta \Rightarrow \boxed{\theta = 10.7^\circ}$$

wave angle at $h = 8m$.

From the constancy of the energy flux we had

$$a_0^2 C_{g_0} \cos \theta_0 = a^2 C_g \cos \theta$$

$$a = a_s \sqrt{\frac{C_g \cos \theta_0}{C_g \cos \theta}}$$

$$C_o = \frac{\lambda_o}{T} = \frac{350}{15} = 23.33 \text{ m/s} \quad C_{S0} = 11.67 \frac{\text{m}}{\text{s}}$$

$$\text{In deep water } C_0 = \frac{1}{2} C_*, \quad \lambda_0 = C_* T,$$

$$\text{At depth } h = 8, \quad c_g = \frac{1}{2} C \left(1 + \frac{2kh}{5h + 2kh} \right), \quad \lambda = C \cdot T. \quad C = \frac{\lambda}{T} = \frac{130}{15} = 8.66 \text{ m/s}$$

$$At \text{ depth } h = 8, \quad C_S = \frac{1}{2} C \left(1 + \frac{2kh}{5h + 2kh} \right), \quad \lambda = C \cdot T. \quad C = \frac{\lambda}{T} = \frac{130}{15} = 8.66 \text{ m/s}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{130} = 0.04833 \text{ rad/m}, \quad h = 8 \text{ m}. \quad C_S = \frac{1}{2} \times 8.66 \times \left(1 + \underbrace{\frac{2 \times 0.04833 \times 8}{5h(2 \times 0.04833 \times 8)}}_{0.90687} \right) = 8.26 \text{ m/s}$$

$$a = \sqrt{\frac{11.67 \times \cos 30^\circ}{8.26 \times \cos 10.7^\circ}} = \sqrt{1.245} = 1.12 \text{ m}$$

the wave amplitude at $h=8\text{ m}$.

