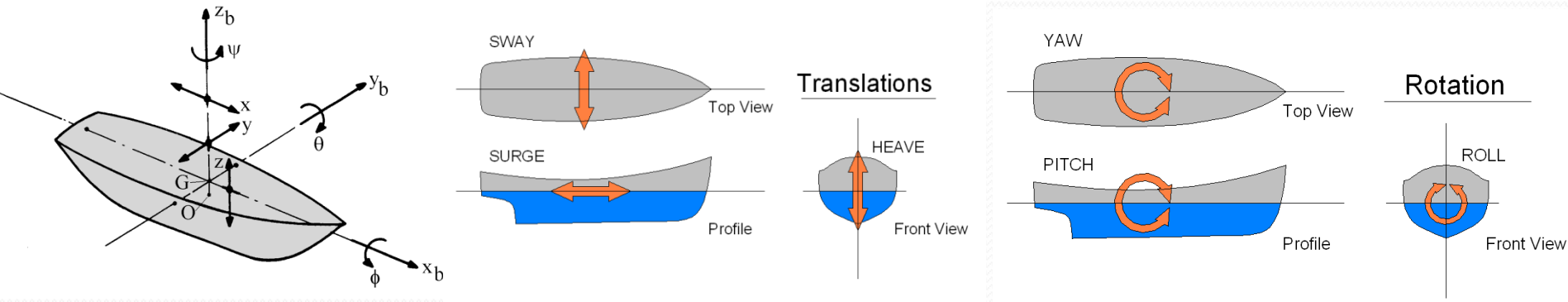# SHIP THEORY - DYNAMICS

SEAKEEPING AND MANEUVERING

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2

Ship Motions

- Ship motions in a seaway are very complicated but can be broken down into 6-degrees of freedom motions relative to 3 mutually perpendicular axes passing through the center of gravity (CG) of ship as shown below.


- **Translational motions:**
1. Surge: Longitudinal disturbance fore & aft along ship's track superimposed on the ship's forward velocity.
2. Sway: Lateral disturbance along the y −axis as port & starboard drift.
3. Heave: Vertical disturbance caused by the imbalance between the weight of the ship and the instantenous changes in the buoyant force resulting from wave action.
- **Rotational motions:**
1. Roll: Transverse oscillatory rotation about ship's transverse axis.
2. Pitch: Longitudinal oscillatory rotation about ship's transverse axis.
3. Yaw: Rotation about ship's vertical axis.

Motion of a Buoy in Waves

- We shall consider the simplest of these motions for the simplest possible geometry; namely, heaving motion of a buoy of circular cross-section in waves.
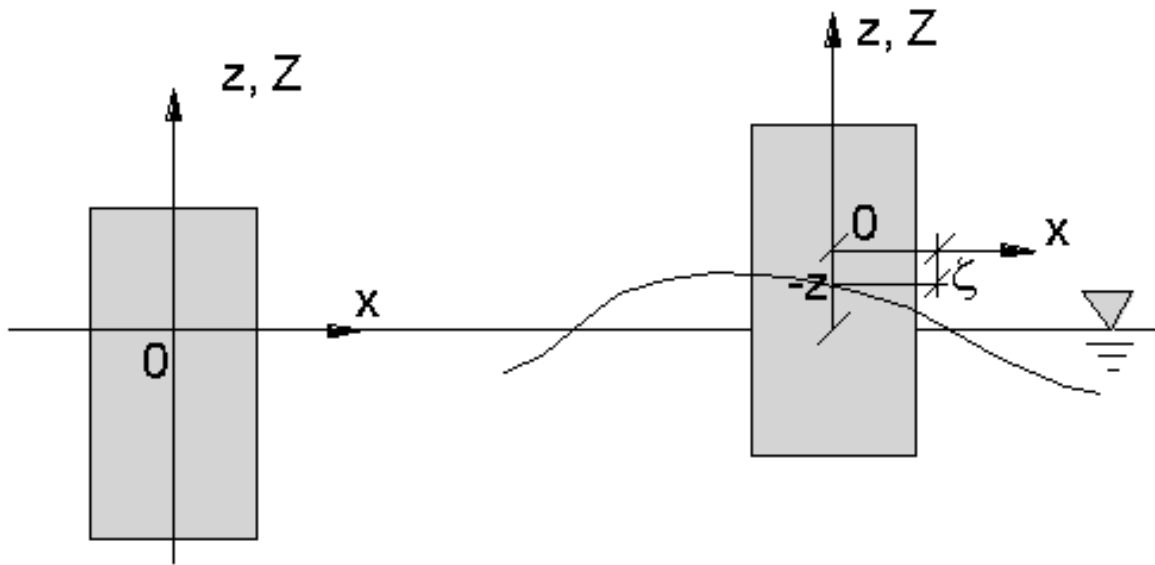
Motion of a Buoy in Waves

- Newton's second law:

$$m\ddot{z} = \sum F_i$$

where the right-hand side is the summation of all vertical forces acting on the buoy, m is the mass, and $\ddot{z} = d^2z/dt^2$ is the instantaneous acceleration.

Supposing that the buoy is subjected to only a steady external force it will float at an equilibrium position with a hydrostatic or buoyancy force of

$$c(-z + \zeta)$$

where z is the vertical displacement of the buoy from its still water level pozition, ζ is the surface elevation, and $c = \rho g A_w$, where A_w is the water plane area. The expression represents the total change in buoyancy from the initial calm-water condition as the result of both change in water level and vertical displacement, neglecting the pressure attenuation with depth. The attenuation correc tion would require that ζ be multiplied by the factor $e^{-k(z+T)}$, where k is the wave number, and T is the draft. If T is large in relation to z, then e^{-kT}.

Motion of Buoy in Waves

- When regular waves are present they generate an exciting force which causes heaving motion. Both z and ζ are harmonic but not necessarily in phase,

$$z = z_0 \cos(\omega t + \varepsilon), \quad \zeta = r \cos(\omega t)$$

where z_0 and r are amplitudes of heave and wave motions, respectively, ω is circular frequency $2\pi/T_w$, T_w wave period, and ε is the phase angle by which heaving motion lags the wave.

Hydrodynamic Forces: One component of hydrodynamic force can be related to relative vertical acceleration between buoy and fluid, and therefore it is 180° out of phase, or opposite to, the buoyancy force,

$$a\left(-\ddot{z} + \ddot{\zeta}\right)$$

where a is the so-called 'added mass' or 'hydrodynamic mass'.

The component of hydrodynamic force which is 90° out of phase with both the relative acceleration and the buoyancy is the damping force:

$$b\left(-\dot{z} + \dot{\zeta}\right)$$

where b is the damping coefficient. Here, the damping force is assumed linear. The approximate attenuation factor e^{-kT} applies to all ζ-related terms.

Motion of a Buoy in Waves

- The equation for dynamic equilibrium of the buoy at any instant, based on Newton's law, can now be stated as follows, without depth attenuation effect

$$m\ddot{z} = F = a\left(-\ddot{z} + \ddot{\zeta}\right) + b\left(-\dot{z} + \dot{\zeta}\right) + c(-z + \zeta)$$

$$\text{Hydrodynamic} \qquad \text{Hydrostatic}$$

After rearranging,

$$(m + a)\ddot{z} + b\dot{z} + cz = a\ddot{\zeta} + b\dot{\zeta} + c\zeta$$

The right-hand side is customarily referred to as the exciting force, which represents the force exerted by the waves on the buoy when it is restrained from vertical motion. Using the previously defined $\zeta = r\cos(\omega t)$,

$$a\ddot{\zeta} + b\dot{\zeta} + c\zeta = r[(c - a\omega^2)\cos(\omega t) - b\omega\sin(\omega t)] = F_1\cos(\omega t) + F_2\sin(\omega t)$$

where $F_1 = r(c - a\omega^2)$ and $F_2 = -rb\omega$. The exiciting force can then be written

$$F = F_1\cos(\omega t) + F_2\sin(\omega t) = F_0\cos(\omega t - \sigma)$$

where $F_0 = \left(F_1{}^2 + F_2{}^2\right)^{1/2} = r[(c - a\omega^2)^2 + (b\omega)^2]^{1/2}$, $\sigma = \tan^{-1}\left(\frac{-b\omega}{c - a\omega^2}\right)$. Here, σ is the phase angle by which the force F lags the wave elevation ζ. We may now consider the depth attenuation factor as follows.

Motion of a Buoy in Waves

- The equation of motion for the buoy in waves

$$(m + a)\ddot{z} + b\dot{z} + cz = F_0 \cos(\omega t - \sigma)\, e^{-kT}$$

Substituting $z = z_a \cos(\omega t + \varepsilon)$ results in

$$z_a = F_0[(c - m\omega^2 - a\omega^2)^2 + b^2\omega^2]^{-1/2}, \qquad \tau = \tan^{-1}\left(\frac{b\omega}{c-(m+a)\omega^2}\right)$$

where τ is the phase angle by which the heaving motion lags the force. To obtain the total phase lag angle ε between the motion and the wave we must take into account the phase angle σ, hence $\varepsilon = -\sigma + \tau$. Using F_0 the heaving motion amplitude z_a is written in the final form of

$$z_a = r\left[\frac{(c - a\omega^2)^2 + b^2\omega^2}{(c - m\omega^2 - a\omega^2)^2 + b^2\omega^2}\right]^{1/2} e^{-kT}$$

and

$$z = z_a \cos(\omega t - \sigma + \tau)$$

In these equations m is known and c is a simple geometrical quantity. The coefficients a and b can be determined by experiment.

Motion of a Buoy in Waves

Consider the free (unforced) motion:

$$(m + a)\ddot{z} + b\dot{z} + cz = 0$$

Letting $z = z_0 e^{i\alpha t}$ results in

$$-(m + a)\alpha^2 + ib\alpha + c = 0$$

which in turn gives for α

$$\alpha_{1,2} = \frac{-ib \pm \sqrt{(ib)^2 - 4[-(m + a)]c}}{-2(m + a)}$$

$$\alpha_{1,2} = i\frac{b}{2(m + a)} \mp \sqrt{\frac{c}{(m + a)} - \frac{b^2}{4(m + a)^2}}$$

Defining $\kappa = b/[c(m + a)]^{1/2}$ as the dimensionless damping parameter

$$n = \frac{b}{2(m + a)} \qquad \omega_d = \omega_n\left(1 - \frac{\kappa^2}{4}\right)^{1/2} \qquad \omega_n = \left(\frac{c}{m + a}\right)^{1/2}$$
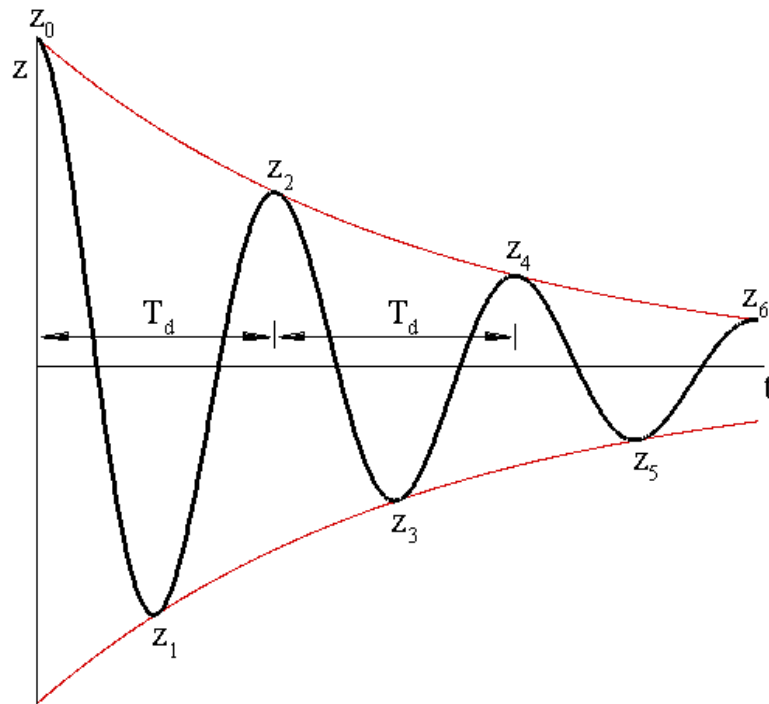
where n is the damping coefficient, ω_n the undamped natural frequency and ω_d is the damped frequency of the system.

Motion of a Buoy in Waves

The solution of the damped unforced system is then
$$z = z_0 e^{-nt} \cos \omega_d t$$

This solution means that the amplitude gradually decreases with time because of damping, as expressed by the factor e^{-nt}.

Motion of a Buoy in Waves

As shown in the figure we may designate the successive amplitudes by z_0, z_1, z_2, ... Then considering the first complete swing, we can obtain the expression for z when $t = T_d \cong T_n$ (for small damping); thus

$$z_2 = z_0 e^{-nT_n}$$

Hence

$$z_2/z_0 = e^{-nT_n}$$

or

$$-\ln(z_2/z_0) = nT_n = \left(\frac{T_n}{2}\right)\left(\frac{b}{m+a}\right)$$

This is referred to as the logarithmic decrement. Substituting $T_n = 2\pi(\frac{m+a}{c})^{1/2}$

$$-\ln(z_2/z_0) = \pi\left(\frac{m+a}{c}\right)^{\frac{1}{2}}\left(\frac{b}{m+a}\right) = \pi\kappa$$

If the damping is linear, $z_4/z_2 = z_2/z_0$ etc., and the samme result will be obtained for each successive circle. Hence, knowing m, a, and T_n, we can solve for the damping coefficient, b. The complete solution to the equation for the buoy in waves can thus be obtained. κ, which is used in the forced solution, can also be computed from the above equation.

Motion of a Buoy in Waves

Now turning back to the forced (due to waves) solution

$$z = z_a \cos(\omega t - \sigma + \tau)$$

$$z_a = r \left[\frac{(c - a\omega^2)^2 + b^2\omega^2}{(c - m\omega^2 - a\omega^2)^2 + b^2\omega^2} \right]^{1/2} e^{-kT}$$

Let us define a nondimensional quantity known as the tuning factor

$$\Lambda = \frac{\omega}{\omega_n} = \omega \left(\frac{m + a}{c} \right)^{1/2}$$

and rearrange z_a as

$$z_a = r \left[\frac{\left(1 - \dfrac{a\omega^2}{c}\right)^2 + \dfrac{b^2\omega^2}{c^2}}{\left(1 - \dfrac{(m + a)\omega^2}{c}\right)^2 + \dfrac{b^2}{c^2}\omega^2} \right]^{1/2} e^{-kT}$$
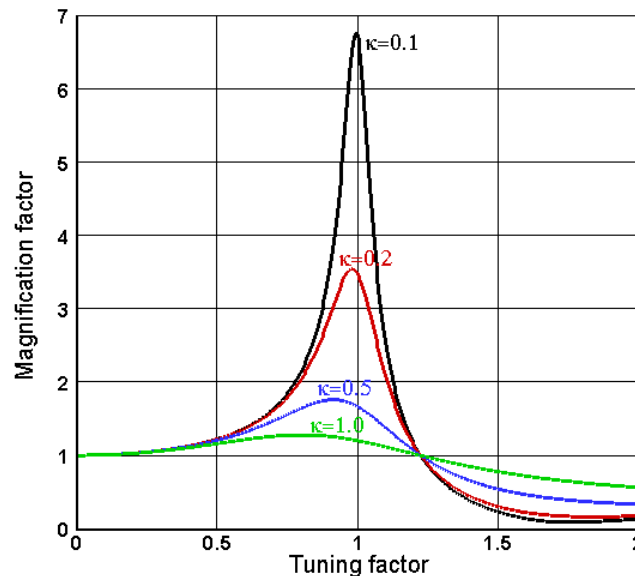
The above amplitude may be expressed in terms of the nondimensional tuning factor Λ and the nondimensional damping parameter κ.

Motion of a Buoy in Waves

Defining a dimesionless ratio μ called the magnification factor, representing the ratio of buoy motion to wave motion at draft T as

$$\mu = \left[\frac{\left(1 - \frac{\Lambda^2}{3}\right)^2 + \kappa^2 \Lambda^2}{(1 - \Lambda^2)^2 + \kappa^2 \Lambda^2} \right]^{1/2}$$

We can then write the solution as $z(t) = \mu r e^{-kT} \cos(\omega t - \sigma + \tau)$.

Unresisted Rolling in Still Water

- **Unresisted rolling in still water:** One of the important motions is rolling. Capsizing of ships mostly occurs in rolling motion in waves. Here, we first consider the rolling in still water without any external forcing. The equation of motion for undamped roll motion is given by

$$I \frac{d^2 \phi}{dt^2} + M = 0$$

where I is the mass moment of inertia of the ship about a longitudinal axis through the center of gravity, M is the righting moment, and ϕ is the angle of inclination of the ship from the vertical. Letting

$$I = \frac{\Delta}{g} k^2$$

where Δ is the displacement, g the gravitational acceleration and k is the radius of gyration of mass of ship about a longitudinal axis through the center of gravity. For small angles of inclination

$$M = \Delta \cdot \overline{GZ} = \Delta \cdot \overline{GM} \cdot \sin \phi = \Delta \cdot \overline{GM} \cdot \phi$$

Unresisted Rolling in Still Water

Substituting these values we have

$$\frac{d^2\phi}{dt^2} + \frac{g\overline{GM}}{k^2}\phi = 0$$

The above equation is the equation for simple harmonic motion having period

$$T_\phi = \frac{2\pi k}{\sqrt{g\overline{GM}}} = \frac{2k}{\sqrt{\overline{GM}}}$$

the latter expression being valid for SI units.

Example: A 10,000 ton-ship has $\overline{GM} = 0.9\ m$ and $T_\phi = 15\ s$. Determine the rolling period after moving 1000 tons symetrically away from a mean distance of 3 m to a mean distance of 6 m.

Solution: Calculate the radius of gyration of mass $k = \frac{T_\phi\sqrt{\overline{GM}}}{2} = \frac{15\sqrt{0.9}}{2} = 7.1\ m$ then $I = \Delta k^2/g = 10000 \cdot 7.1^2 = 504100\ ton \cdot m^2$. The altered mass moment of inertia $I' = 504100 + 1000(6^2 - 3^2) = 531100\ ton \cdot m^2$. The new radius of gyration of mass $k' = \sqrt{\frac{531100}{10000}} = 7.29\ m$ and the rolling period $T_\phi{'} = \frac{2\cdot7.29}{\sqrt{0.9}} = 15.4\ s$.

Unresisted Rolling in Still Water

Solving the differential equation

$$\frac{d^2\phi}{dt^2} + \frac{g\overline{GM}}{k^2}\phi = 0$$

gives

$$\phi(t) = \left(\frac{d\phi_A}{dt}\right)\left(\frac{T_\phi}{2\pi}\right)\sin\left(\frac{2\pi t}{T_\phi}\right) + \phi_A\cos\left(\frac{2\pi t}{T_\phi}\right)$$

$$\phi(t) = \left(\frac{d\phi_A}{dt}\right)\frac{1}{\omega_\phi}\sin\omega_\phi t + \phi_A\cos\omega_\phi t$$

where at $t = 0$, $\phi = \phi_A$ and $d\phi/dt = d\phi_A/dt$ are the initial angle of roll and the initial angular velocity of roll while $\omega_\phi = 2\pi/T_\phi$ is the circular frequency. If we assign as initial conditions that $\phi_A = 0$ and $d\phi/dt = d\phi_A/dt$ when $t = 0$,

$$\phi(t) = \left(\frac{d\phi_A}{dt}\right)\frac{1}{\omega_\phi}\sin\omega_\phi t$$

whereas if, when $t = 0$, the inclination is equal to ϕ_A and $d\phi_A/dt = 0$

$$\phi(t) = \phi_A\cos\omega_\phi t$$

Unresisted Rolling Among Waves

- **Unresisted rolling among waves:** To a first approximation the wave-disturbing moment is proportional to the wave slope $\tan \alpha_M = 2\pi \zeta_a L_w$ For small angles α_M in radians may be substituted for $\tan \alpha_M$ hence

$$I \frac{d^2\phi}{dt^2} + \Delta \overline{GM}\phi = \Delta \overline{GM}(2\pi \zeta_a / L_w) \sin \omega_w t$$

where ζ_a is the wave amplitude, L_w the wavelength, and ω_w the wave frequency. The above equation may be re-written in the following form

$$\frac{d^2\phi}{dt^2} + {\omega_\phi}^2 = {\omega_\phi}^2 \, \alpha_M \sin \omega_w t$$

where the right-hand side is the exciting moment. Solving the above differential equation with initial conditions that $\phi_A = 0$ and $d\phi_A/dt = 0$ gives

$$\phi(t) = \frac{\alpha_M}{1 - \frac{{T_\phi}^2}{{T_w}^2}} \left(\sin \omega_w t - \frac{T_\phi}{T_w} \sin \omega_\phi t \right)$$

Note that when $T_w = T_\phi$ the equation reduces to 0/0, which must be evaluated for the limit to get $\phi(t) = (\alpha_M/2)(\sin \omega_w t - \omega_w t \cos \omega_w t)$.

Resisted Rolling in Still Water

- **Resisted rolling in still water:** If the resistance to rolling is $A(d\phi/dt)$,

$$I\frac{d^2\phi}{dt^2} + A\frac{d\phi}{dt} + \Delta\overline{GM}\phi = 0$$

The solution of the above equation yields the following results

$$A = \frac{\Delta T_\phi \overline{GM} K_1}{\pi^2}, \qquad T_\phi{}' = \frac{T_\phi}{\left(1 - \dfrac{K_1{}^2}{\pi^2}\right)^{1/2}}$$

where K_1 is a coefficient related to A by the above relationship and is less than unity; therefore $K_1{}^2/\pi^2$ is less than 0.1 and the denominator of $T_\phi{}'$ is less than unity. Consequently, the period of resisted rolling differs from the period of unresisted rolling only by a small amount. For instance, let $K_1 = 0.1$ then

$$\left(1 - \frac{K_1{}^2}{\pi^2}\right)^{1/2} = \sqrt{0.999} = 0.999, \qquad T_\phi{}' = \frac{T_\phi}{0.999} = 1.0005 T_\phi$$