Coastal Engineering 109 (2016) 128–133

Contents lists available at ScienceDirect

Coastal Engineering

journal homepage: www.elsevier.com/locate/coastaleng

Improved Korteweg & de Vries type equation with consistent shoaling characteristics

S. Beji *

Faculty of Naval Architecture and Ocean Engineering, Istanbul Technical University, Maslak, 34469 Istanbul, Turkey

ARTICLE INFO

Article history:
Received 23 June 2015
Received in revised form 4 October 2015
Accepted 23 October 2015
Available online 27 November 2015

Keywords:
Kortweg & de Vries equation
Nonlinearity
Dispersion
Shoaling
Varying water depths

ABSTRACT

An improved Korteweg & de Vries type equation for uneven water depths with consistent linear shoaling characteristics is derived. The improvement of the equation is with respect to linear dispersion characteristics while consistency in linear shoaling characteristics is achieved via an exact agreement of the shoaling rate of the wave equation with that obtained from the principle of energy flux concept. Improvements in both linear dispersion and linear shoaling properties are demonstrated analytically and numerically by simulating a challenging experimental test case of nonlinear wave propagation over a submerged bar.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

The historic observation of John Scott Russell on horseback of a solitary wave in 1834 and his subsequent experiments in 1845 spurred the works on mathematical description of solitary waves (Miles, 1980). Boussinesq (1872) was the first to develop a wave equation with solitary wave solution. Lord Rayleigh (1876) made quite a similar derivation to the same purpose but in closing acknowledged the priority of Boussinesq's work. Korteweg and de Vries (1895) presented what might be termed the one-directional form of Boussinesq's one-dimensional wave model and showed that the equation admitted not only of solitary waves but also of a new class of permanent periodic waves named “cnoidal” waves as solution.

For more than half a century, till the early 1960s, the subject of solitary waves was quite dormant. Then, especially with the advent of computers, the interest in the Korteweg & de Vries equation or the KdV equation began growing. Miles (1981) gives a very illustrative graph of the number of citations of Korteweg and de Vries (1895) by year. Meanwhile, two dimensional forms of Boussinesq equations for varying bottom topography were derived first by Mei and Le Méhauté (1966) using the bottom velocity as the dependent variable and shortly afterwards by Peregrine (1967) using the averaged velocity instead of the somewhat ambiguous bottom velocity. These derivations were important for practical applications in coastal regions. In particular, Peregrine's Boussinesq equations for varying bathymetry have become almost the standard Boussinesq model of the coastal engineering community. Beginning from the 1970s Abbott and co-workers have developed numerical schemes for solving one- and two-dimensional wave propagation problems via Boussinesq models (Abbott, 1974; Abbott et al., 1973, 1978, 1984).

Witting (1984) made an outstanding contribution by introducing an improvement to the dispersion characteristics of Boussinesq type equations by means of a *new velocity* variable. At the same time his numerical treatment of the one dimensional equations included all the nonlinear terms though he rightly pointed out that the equations could not be called fully-nonlinear as the series expansion in vertical coordinate necessarily contained only finite number of terms. That is to say, the limited order of dispersion terms consequently limits the order of nonlinearity. This important point seems to be overlooked in some subsequent publications which claim full nonlinearity in Boussinesq equations, which in essence is not possible.

Madsen et al. (1991) added second-order terms to Boussinesq equations to improve the dispersion characteristics. Later, Madsen and Sørensen (1992) extended the improved equations to varying bathymetry. In the same vein, Beji and Nadaoka (1996) introduced the concept of partial replacement rather than addition and claimed to derive a consistent model, basing their arguments on the constancy of energy flux.

Nwogu (1993) gave an alternative derivation again of the Boussinesq equations with better dispersion properties by expressing the equations in terms of a velocity at an arbitrary water depth. Although never noticed the approach of Nwogu was indeed equivalent to that of Witting: Nwogu used the *velocity at an arbitrary depth* while Witting used a

* Tel.: +90 212 285 6442; fax: +90 212 285 6454.
E-mail address: sbeji@itu.edu.tr.

http://dx.doi.org/10.1016/j.coastaleng.2015.10.009
0378-3839/© 2015 Elsevier B.V. All rights reserved.

new velocity expressed in terms of unknown coefficients. Nwogu's derivation is weakly nonlinear and for 2-D whereas Witting's derivation is strongly nonlinear and for 1-D; otherwise the two derivations stem from the same argument, which is essentially to use a different horizontal velocity variable other than a conventional one such as the mean velocity or the surface velocity.

With regard to the KdV equation a notable contribution was made by Benjamin et al. (1972). In a very formal and thorough analysis they showed that the term with three spatial derivatives representing the dispersion might be replaced by a term comprising two spatial derivatives and a time derivative. While the new form of the equation has the same formal justification it has mathematical and computational advantages over the former. Remarkably, before the formal justification of Benjamin et al. (1972), Peregrine (1966) used exactly the same form of the KdV equation in his numerical calculations of an undular bore. In time, replacing the nonlinear or dispersion terms (i. e. the second-order terms) with their equivalents has become a usual practice; Mei (1989 p. 550) enumerates two different nonlinear and four different dispersive terms, which in turn generate eight different KdV type equations.

In this work first a KdV type equation with mixed dispersion terms is derived from the combined form of the improved Boussinesq equations given by Beji and Nadaoka (1996). The improved Boussinesq equations are based on the application of partial replacement technique to the classical equations of Peregrine (1967) for varying water depth. Following the derivation of a generalized KdV equation with mixed dispersion and linear shoaling terms the linear shoaling gradient of the equation is compared with that obtained from the energy flux concept. Such direct comparability clearly indicates an intricate and inseparable link between the dispersion and the shoaling terms. Accordingly then the form of the KdV equation corresponding to Peregrine's classical Boussinesq model is found to produce a shoaling gradient in complete agreement with the energy flux concept. Though based on the unimproved Boussinesq equations the new type KdV equation possesses mixed dispersion and shoaling terms with improved characteristics. A numerical example based on the simulation of an experiment of Beji and Battjes (1994) is given to demonstrate the improved dispersion and shoaling aspects of the new equation.

2. Combined form of improved Boussinesq equations

By introducing the partial replacement technique to Peregrine's (1967) Boussinesq model for varying water depths Beji and Nadaoka (1996) gave the following continuity and momentum equations:

$$\eta_t + \nabla\cdot[(h+\eta)\mathbf{u}] = 0 \tag{1}$$

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + g\nabla\eta = \frac{(1+\beta)}{2}h\nabla[\nabla\cdot(h\mathbf{u}_t)] + \frac{\beta}{2}gh\nabla[\nabla\cdot(h\nabla\eta)] - \frac{(1+\beta)}{6}h^2\nabla(\nabla\cdot\mathbf{u}_t) - \frac{\beta}{6}gh^2\nabla\left(\nabla^2\eta\right) \tag{2}$$

where $\mathbf{u}$ is the vertically averaged or mean horizontal velocity vector with components (u, v) and η is the free surface displacement as measured from the still water level. $h = h(x, y)$ is the spatially varying local water depth and g is the gravitational acceleration. ∇ stands for two-dimensional horizontal gradient operator with components $(\partial/\partial x, \partial/\partial y)$ while subscript t denotes partial differentiation with respect to time. β is a non-dimensional scalar determined according to Padé approximation of the linear theory dispersion relation so that the resulting equations have better dispersion characteristics. Degenerate cases $\beta = 0$ and $\beta = -1$ indicate respectively Peregrine's original momentum equation and a full replacement of the dispersion term in the momentum equation.

The above equations, though nonlinear, may be combined to result in a single equation in terms of the free surface displacement η by appropriate approximations, which are not repeated here for the sake of brevity. The resulting equation is

$$\eta_{tt} = gh\nabla^2\eta + \frac{(1+\beta)}{3}h^2\nabla^2\eta_{tt} - \frac{\beta}{3}gh^3\nabla^2\left(\nabla^2\eta\right) + \frac{3}{2}g\nabla^2(\eta^2) + g\nabla h\cdot\nabla\eta + (1+\beta)h\nabla h\cdot\nabla\eta_{tt} - 2\beta gh^2\nabla h\cdot\nabla\left(\nabla^2\eta\right) \tag{3}$$

in which only the terms containing the first spatial derivative of the depth are retained. Truncation of higher depth gradients implies the use of *mild-slope* approximation, which is maintained throughout the work wherever necessary. 1-D form of Eq. (3) reads

$$\eta_{tt} = gh\eta_{xx} + \frac{(1+\beta)}{3}h^2\eta_{xxtt} - \frac{\beta}{3}gh^3\eta_{xxxx} + \frac{3}{2}g(\eta^2)_{xx} + gh_x\eta_x + (1+\beta)hh_x\eta_{xtt} - 2\beta gh^2h_x\eta_{xxx}. \tag{4}$$

The degenerate case $\beta = -1$ for constant depth gives the original derivation of Boussinesq (1872):

$$\eta_{tt} = gh\eta_{xx} + \frac{1}{3}gh^3\eta_{xxxx} + \frac{3}{2}g(\eta^2)_{xx}. \tag{5}$$

It is worthwhile to point out that Boussinesq's entire work was based on Eq. (5) (his Eq. (26)) and that he never gave his model separately as continuity and momentum equations.

3. Improved KdV type equation for varying depth

An improved KdV-like equation for uneven bathymetry is now derived. Derivation is based on the combined 1-D Boussinesq model, Eq. (4). First, introduce a co-ordinate system moving in the positive $x-$ direction with the non-dispersive phase velocity $C = \sqrt{gh}$ so that the evolutions of the wave form in this moving system is slow, permitting to write the following new co-ordinates:

$$\sigma = x - Ct, \qquad \tau = \varepsilon t \tag{6}$$

where ε is a small parameter indicating the weak changes of the wave form in time in the moving co-ordinate system. Expressing the terms in Eq. (4) in the new co-ordinate system gives

$$\begin{aligned}&\eta_{xx} = \eta_{\sigma\sigma}, \qquad \eta_{xxx} = \eta_{\sigma\sigma\sigma}, \qquad \eta_{xxxx} = \eta_{\sigma\sigma\sigma\sigma}\\&\eta_{tt} = C^2\eta_{\sigma\sigma} - 2\varepsilon C\eta_{\sigma\tau} + \varepsilon CC_\sigma\eta_\sigma, \quad \eta_{xtt} = C^2\eta_{\sigma\sigma\sigma} - 2\varepsilon C\eta_{\sigma\sigma\tau} + \varepsilon CC_\sigma\eta_{\sigma\sigma}\\&\eta_{txtx} = C^2\eta_{\sigma\sigma\sigma\sigma} - 2\varepsilon C\eta_{\sigma\sigma\sigma\tau} + 4\varepsilon CC_\sigma\eta_{\sigma\sigma\sigma} - 3\varepsilon C_\sigma\eta_{\sigma\sigma\tau}\end{aligned} \tag{7}$$

where the terms containing the spatial derivative of C have also been labeled by ε to indicate they are an order higher, and the terms proportional to ε^2 are all neglected. The neglect of these terms probably causes the resulting KdV type equations to lose the energy conservation characteristics that the Boussinesq equations possess. While the Boussinesq model of Beji and Nadaoka (1996) is consistent for any β values, the KdV type model derived from it is consistent for only $\beta = 0$ as is shown in Section 5. Substituting the expressions of Eq. (7) into Eq. (4) and re-arranging results in

$$-\varepsilon 2C\eta_{\sigma\tau} - \frac{1}{3}C^2h^2\eta_{\sigma\sigma\sigma\sigma} + \varepsilon\frac{2(1+\beta)}{3}Ch^2\eta_{\sigma\sigma\sigma\tau} - \frac{3}{2}g(\eta^2)_{\sigma\sigma} + \varepsilon CC_\sigma\eta_\sigma - \varepsilon gh_\sigma\eta_\sigma - \varepsilon\frac{(5-\beta)}{3}C^2hh_\sigma\eta_{\sigma\sigma\sigma} + \varepsilon\frac{5(1+\beta)}{2}Chh_\sigma\eta_{\sigma\sigma\tau} = 0 \tag{8}$$

where the last four terms are the so-called linear shoaling terms. The first two of these four terms originate from the continuity equation and may be put into the same form, and are kinematic in essence; while the last two terms originate from the dispersive terms or the terms related to the non-hydrostatic pressure, which in turn are related to the vertical velocity, and are essentially dynamic in nature. These last two terms are strictly dictated by the linear dispersion terms, η_{xxtt} and η_{xxxx}, and cannot be added or subtracted arbitrarily on the premise of being small; however, they may be manipulated to some degree within allowable limits as have been done above by changing the order of differentiations hence their relative coefficients but not their total value in the transformed expressions.

Before performing an integration with respect to σ the terms $CC_\sigma\eta_\sigma$ and $gh_\sigma\eta_\sigma$ are combined by employing $C^2 = gh$ hence $2CC_\sigma = gh_\sigma$ and the equation is divided by $-2C^{5/2}$ so that it becomes

$$\varepsilon C^{-3/2}\eta_{\sigma\tau} + \frac{1}{6}C^{-1/2}h^2\eta_{\sigma\sigma\sigma\sigma} - \varepsilon\frac{(1+\beta)}{3}C^{-3/2}h^2\eta_{\sigma\sigma\sigma\tau} + \frac{3}{4}gC^{-5/2}\left(\eta^2\right)_{\sigma\sigma} + \varepsilon\frac{1}{4}gC^{-5/2}h_\sigma\eta_\sigma + \varepsilon\frac{(5-\beta)}{6}C^{-1/2}hh_\sigma\eta_{\sigma\sigma\sigma} - \varepsilon\frac{5(1+\beta)}{4}C^{-3/2}hh_\sigma\eta_{\sigma\sigma\tau} = 0. \tag{9}$$

The reason for performing the integration after dividing by $C^{5/2}$ is to lead the equation to a form with consistent linear shoaling characteristics. Such a manipulation changes only the relative values of the shoaling terms but not their total value. Since the non-dispersive phase speed C is a spatially varying function, the integration of Eq. (9) is performed according to the following expressions.

$$\int\left(C^{-1/2}h^2\eta_{\sigma\sigma\sigma\sigma}\right)d\sigma = C^{-1/2}h^2\eta_{\sigma\sigma\sigma} - \frac{7}{4}C^{-1/2}hh_\sigma\eta_{\sigma\sigma}$$
$$\int\left(C^{-3/2}h^2\eta_{\sigma\sigma\sigma\tau}\right)d\sigma = C^{-3/2}h^2\eta_{\sigma\sigma\tau} - \frac{5}{4}C^{-3/2}hh_\sigma\eta_{\sigma\tau}. \tag{10}$$

The validity of the above equalities, correct to the second spatial derivative of C and h, may be verified easily by differentiating both sides with respect to σ. The remaining terms are integrated by simply removing a subscript σ since they are all second-order and the contribution of higher-order terms are all neglected. The integrated and re-arranged form of Eq. (9) is then

$$\varepsilon\eta_\tau + \frac{1}{6}Ch^2\eta_{\sigma\sigma\sigma} - \varepsilon\frac{(1+\beta)}{3}h^2\eta_{\sigma\sigma\tau} + \frac{3}{4}Ch^{-1}\left(\eta^2\right)_\sigma + \varepsilon\frac{1}{4}Ch^{-1}h_\sigma\eta + \varepsilon\frac{(13-4\beta)}{24}Chh_\sigma\eta_{\sigma\sigma} - \varepsilon\frac{5(1+\beta)}{6}hh_\sigma\eta_{\sigma\tau} = 0. \tag{11}$$

In order to go back to the original co-ordinate system, the following inverse transformations are needed

$$\varepsilon\eta_\tau = \eta_t + C\eta_x, \quad \varepsilon\eta_{\tau\sigma\sigma} = \eta_{xxt} + C\eta_{xxx} + 2C_x\eta_{xx}$$
$$\varepsilon\eta_{\sigma\tau} = \eta_{xt} + C\eta_{xx}, \quad \eta_{\sigma\sigma} = \eta_{xx}, \quad \eta_{\sigma\sigma\sigma} = \eta_{xxx} \tag{12}$$

Substituting Eq. (12) into Eq. (11) results in a generalized KdV type equation for uneven depths:

$$\eta_t + C\eta_x - \frac{(1+2\beta)}{6}Ch^2\eta_{xxx} - \frac{(1+\beta)}{3}h^2\eta_{xxt} + \frac{3}{4}Ch^{-1}\left(\eta^2\right)_x + \frac{1}{4}Ch^{-1}h_x\eta - \frac{(15+32\beta)}{24}Chh_x\eta_{xx} - \frac{5(1+\beta)}{6}hh_x\eta_{xt} = 0. \tag{13}$$

The above KdV type equation embodies all the known KdV type equations as special cases. First of all since all available KdV-like equations are basically formulated for constant depth, the terms proportional to h_x must be dropped. Then, setting $\beta = -1$ gives the original Korteweg and de Vries (1895) equation while $\beta = -1/2$ gives the so-called regularized KdV equation of Benjamin et al. (1972) or of Peregrine (1966). In Section 5 it is shown that $\beta = 0$, derivation based on the original equations of Peregrine (1967), results in a model in exact agreement with the constancy of energy flux. The last two terms of Eq. (13) are the terms that ensure consistent linear shoaling properties.

4. Dispersion characteristics

Dispersion characteristics of the generalized KdV type equation are now examined. With a_0 denoting the constant wave amplitude, substituting $\eta = a_0 \exp[ik(x - C_K t)]$ into the linearized, constant-depth form of Eq. (13) and solving for the phase velocity C_K gives

$$C_K = C\left(\frac{1 + \frac{1}{6}(1+2\beta)k^2h^2}{1 + \frac{1}{3}(1+\beta)k^2h^2}\right) \tag{14}$$

where, as indicated before, $C = \sqrt{gh}$ is the non-dispersive shallow water wave celerity, k is the wave number, and C_K is the phase velocity of the generalized KdV type equation. On the other hand, the "exact" phase speed according to the linear theory or the second-order Stokes theory is given by

$$C_E = C\sqrt{\frac{\tanh\,kh}{kh}} \tag{15}$$

Eq. (15) may be expressed in terms of Padé approximations or Padé *approximants*, as named specifically (Baker and Graves-Morris, 1980). First expanding tanh kh in a Maclaurin series, then dividing the expansion by kh and taking the square root of it, and finally establishing the [2/2] Padé approximant gives

$$C_P = C\left(\frac{1 + \frac{9}{60}k^2h^2}{1 + \frac{19}{60}k^2h^2}\right) + O\left(k^6h^6\right). \tag{16}$$

In order for Eq. (14) be identical with Eq. (16), the free dispersion parameter β must be chosen as $\beta = -1/20$. However, in Section 5 it is shown that β must be set to zero if a consistent model is required. In Fig. 1, the accuracy of the dispersion relationship (14) is compared with the exact expression (15) for $\beta = 0$, $\beta = -1/20$, $\beta = -1/2$, and $\beta = -1$ by depicting the error percentage $100(C_K - C_E)/C_E$ versus the dispersion parameter kh or relative depth. As the figure shows, $\beta = -1/20$, the case corresponding to the [2/2] Padé approximant, has the lowest error percentage compared with the rest. It has 6.8 % error for $kh = \pi$, which is basically the deep water limit. $\beta = 0$ is somewhat inferior to $\beta = -1/20$ with 9.5 % error for $kh = \pi$. The reason lies in the fact that while the dispersion relation for $\beta = -1/20$ corresponds to a representation of the exact dispersion relationship correct to $O(k^6h^6)$, as obtained by establishing the [2/2] Padé approximant, the dispersion relation for $\beta = 0$ corresponds to a representation correct to $O(k^4h^4)$. However; depending on the judgment, $\beta = 0$ may be preferred in favor of using a consistent model. On the other hand, the regularized KdV equation $\beta = -1/2$ and the classical KdV equation

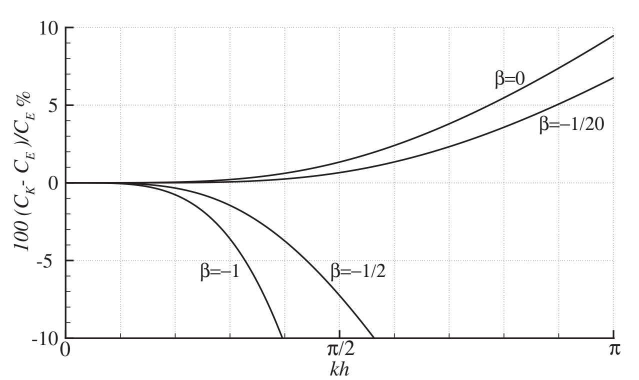

Fig. 1. Relative error percentage $100(C_K - C_E)/C_E$ versus relative depth kh for different forms of KdV equation.

$\beta = -1$ have respectively -32.9 % and -214.5 % relative errors for $kh = \pi$. These errors, being quite large, exceed the set limits of the graph and therefore cannot be observed from the figure.

5. A consistent model

A wave model with consistent linear shoaling characteristics is now established by comparing the linear shoaling gradient obtained from the wave equation with that obtained from the energy flux concept. Eq. (13) in linearized form is now re-written as

$$\eta_t + C\eta_x - pCh^2\eta_{xxx} - qh^2\eta_{xxt} + (C/4h)h_x\eta - rChh_x\eta_{xx} - shh_x\eta_{xt} = 0 \tag{17}$$

where the non-dimensional coefficients $p = (1+2\beta)/6$, $q = (1+\beta)/3$, $r = (15+32\beta)/24$, $s = 5(1+\beta)/6$ are used for the sake of a simpler notation. The shoaling gradient; that is, the spatial change of wave amplitude due to change in depth is obtained by first using the wave model itself and then by invoking the constancy of energy flux.

5.1. Shoaling gradient from wave model

Let $\eta(x,t) = a(x)\exp[i(\omega t - \varphi(x))]$ with $a(x)$ representing the spatially varying wave amplitude and $\varphi(x)$ the phase function such that $d\varphi(x)/dx = k(x)$. Note that both the wave amplitude and the wave number are spatially varying quantities. Substituting $\eta(x,t)$ into Eq. (17) and keeping only the first spatial derivatives of $k(x)$ results in

$$\begin{aligned}&\left(C + 3pCk^2h^2 - 2q\omega kh^2\right)a_x + \left(C/4h + rCk^2h - s\omega kh\right)ah_x\\&+\left(3pCkh^2 - q\omega h^2\right)ak_x + \left(\omega - kC + q\omega k^2h^2 - pCk^3h^2\right)ai = 0.\end{aligned} \tag{18}$$

The real and imaginary parts must vanish separately. The last term, the imaginary part, is the linear dispersion relationship of the wave model and is indeed identical with Eq. (14) when ω is set to kC_K. Thus, setting the imaginary part to zero gives $\omega = kC(1+pk^2h^2)/(1+qk^2h^2)$ which, by recalling $\omega_x = 0$, may be differentiated with respect to x to obtain an expression between k_x and h_x:

$$\frac{k_x}{k} = -\frac{1}{2}\left(\frac{1+(5p-3q)k^2h^2+pqk^4h^4}{1+(3p-q)k^2h^2+pqk^4h^4}\right)\frac{h_x}{h} \tag{19}$$

Using the expression for ω and Eq. (19) in the real part of Eq. (18) results in a relationship between the amplitude gradient a_x and the depth gradient h_x as dictated by the wave equation itself:

$$\begin{aligned}&\frac{a_x}{a} = -\frac{1}{4}\left(\frac{\Gamma_1}{\Gamma_0^2}\right)\frac{h_x}{h}\\&\Gamma_0 = 1+(3p-q)k^2h^2+pqk^4h^4\\&\Gamma_1 = 1-(3p-2q-4r+4s)k^2h^2-(30p^2+7q^2-28pq-12pr+16ps-4qs)k^4h^4\\&\qquad -(26p^2q-15pq^2+12p^2s+4q^2r-16pqr)k^6h^6-4pq(pq-qr+ps)k^8h^8.\end{aligned} \tag{20}$$

5.2. Shoaling gradient from energy flux concept

A relationship between a_x and h_x is now derived by invoking the constancy of energy flux; that is, $\partial(a^2C_g)/\partial x = 0$. The group velocity dictated by the dispersion relationship of Eq. (17) is obtained as

$$C_g = \frac{d\omega}{dk} = C\left(\frac{1+(3p-q)k^2h^2+pqk^4h^4}{\left(1+qk^2h^2\right)^2}\right). \tag{21}$$

Differentiating C_g with respect to x and making use of Eq. (19) in $a_x/a = -(1/2)(C_g)_x/C_g$ gives

$$\begin{aligned}&\frac{a_x}{a} = -\frac{1}{4}\left(\frac{\Gamma_2}{\Gamma_0^2}\right)\frac{h_x}{h}\\&\Gamma_0 = 1+(3p-q)k^2h^2+pqk^4h^4\\&\Gamma_2 = 1+4(3p-2q)k^2h^2+3(5p^2+q^2-4pq)k^4h^4+4p^2qk^6h^6+p^2q^2k^8h^8\end{aligned} \tag{22}$$

which is the expression dictated by the constancy of energy flux according to the dispersion relationship of the wave equation. Note that unlike Eq. (20), Eq. (22) does not contain the parameters r and s which arise from the linear shoaling terms of the wave equation. However, they must be related to p and q for a consistent model if Eq. (20) be identical with Eq. (22). Such a relationship reveals the intricate connection between the dispersion terms and the shoaling terms. A careful examination of the derivation process shows that the shoaling terms, terms proportional to r and s, originate from the dispersion terms, terms proportional to p and q. Therefore, they cannot be treated as independent parameters and must be related.

5.3. Matching of two shoaling gradients

Requiring Eq. (20) be identical with Eq. (22) in turn requires $\Gamma_1 = \Gamma_2$ as the denominator Γ_0^2 is the same for both expressions. Equating term by term requires the following equalities be satisfied.

$$q = 2p, \qquad r = 15p/4, \qquad s = 5p \tag{23}$$

in which p is free to take on any value. Recalling now $p = (1+2\beta)/6$, $q = (1+\beta)/3$, $r = (15+32\beta)/24$, $s = 5(1+\beta)/6$ it is seen at once that the set of equalities in Eq. (23) is satisfied if and only if $\beta = 0$ so that $p = 1/6$, $q = 1/3$, $r = 5/8$, $s = 5/6$. The solution is unique as no other choice is possible for β; neither the original $\beta = -1$ nor the regularized KdV equation $\beta = -1/2$ complies with equalities of Eq. (23). Likewise, if p is selected according to the [2/2] Padé approximation as $p = 9/60$, Eq. (23) would require $q = 2p = 18/60$ which is at variance with $q = 19/60$ of Eq. (16), albeit slightly.

It is worthwhile to point out that when $p = q = r = s = 0$, the equation degenerates into the combined unidirectional form of the non-dispersive shallow water equations and the shoaling gradient becomes $\Gamma_1/\Gamma_0^2 = \Gamma_2/\Gamma_0^2 = 1$ hence $a_x/a = -h_x/4h$, which is Green's law of shoaling for a canal of gradually varying depth but constant breadth (Lamb, 1932; page 275). The term $(C/4h)h_x\eta$ appearing in Eq. (13) is responsible for this result; as indicated before it is the geometric part of shoaling, originating from the continuity equation. The shoaling terms proportional to r and s originate from the dispersive terms of the momentum equation.

Fig. 2 depicts Γ_1/Γ_0^2 and Γ_2/Γ_0^2 for $\beta = 0$ and $\beta = -1/20$ as well as the exact ratio $(\Gamma_2/\Gamma_0^2)_E$ obtained using the group velocity corresponding to exact dispersion relationship. The consistent model $\beta = 0$ produces identical results; that is, Γ_1 and Γ_2 are the same. On the other hand, as shown in the figure, they differ for $\beta = -1/20$. Despite the existence of two different gradients, both shoaling gradients for $\beta = -1/20$ approximate the exact expression better and that the difference between Γ_1 and Γ_2 becomes appreciable only for high kh values. Compared to dispersion characteristics shown in Fig. 1, Fig. 2 reveals less accurate shoaling characteristics with increasing relative depth. This is simply because the shoaling gradient is a higher order expansion compared to the dispersion relation hence diverges rapidly from the exact shoaling gradient. Finally, it must be emphasized that prediction accuracy of a wave model in shoaling region is greatly dependent on its shoaling terms; Simarro (2013) introduced a different concept of analysis based on energy balance to investigate the shoaling characteristics of Boussinesq type equations. In this work, the concept of energy flux is employed for producing a consistent model and the performance of

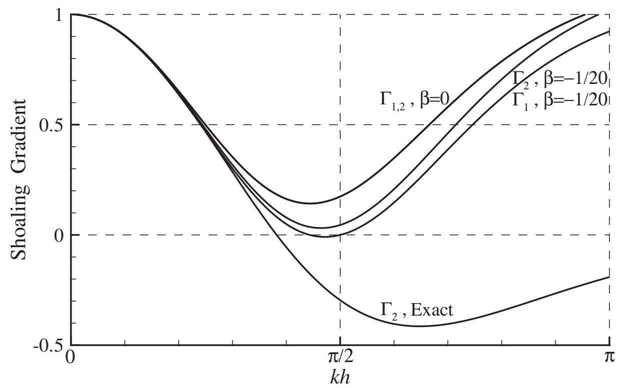

Fig. 2. Shoaling gradient coefficients Γ_1/Γ_0^2 and Γ_2/Γ_0^2 versus relative depth kh. Labels in the figure show only numerators Γ_1, Γ_2 for briefness.

wave model in an experimental simulation for different β values is investigated in sub-section 6.1.

6. Numerical treatment

Various numerical methods may be used for the numerical solution of Eq. (13); particularly a finite-difference scheme would be preferable. The existence of the third spatial derivative of surface elevation however necessitates a more sophisticated discretization than a straightforward finite-difference formulation if a robust scheme is aimed. Such a scheme is planned for a future work as an extension of the present improved KdV model to an improved Kadomtsev–Petviashvili or briefly the KP type equation (Kadomtsev and Petviashvili, 1970). Here, the spectral method as detailed in Beji and Nadaoka (1999) for a unidirectional wave model is directly adopted to Eq. (13). The procedure is originally due to Bryant (1973); Freilich and Guza (1984) applied a similar approach to variants of Boussinesq equations while Madsen and Sørensen (1993) used the technique in a thorough investigation of their improved Boussinesq model. Letting $\eta(x,t)=\sum_{n=-\infty}^{+\infty}A_n(x)e^{i[\omega_n t-\int k_n(x)dx]}$ in Eq. (13) and then changing to the real variables by setting $A_n(x)=\frac{1}{2}[a_n(x)-ib_n(x)]$ gives the following evolution equations for the real variables $a_n(x)$ and $b_n(x)$

$$\frac{da_n}{dx}=-\frac{\alpha_s}{\alpha_1}a_n+\frac{3C}{4h}\sum_{m=1}^{N-n}\alpha^+\left[(a_m b_{n+m}-a_{n+m}b_m)\cos\theta^+ +(a_m a_{n+m}+b_m b_{n+m})\sin\theta^+\right]$$
$$+\frac{3C}{8h}\sum_{m=1}^{n-1}\alpha^-\left[(a_m b_{n-m}+a_{n-m}b_m)\cos\theta^- +(a_m a_{n-m}-b_m b_{n-m})\sin\theta^-\right] \tag{24}$$

$$\frac{db_n}{dx}=-\frac{\alpha_s}{\alpha_1}b_n-\frac{3C}{4h}\sum_{m=1}^{N-n}\alpha^+\left[(a_m a_{n+m}+b_m b_{n+m})\cos\theta^+ -(a_m b_{n+m}-a_{n+m}b_m)\sin\theta^+\right]$$
$$-\frac{3C}{8h}\sum_{m=1}^{n-1}\alpha^-\left[(a_m a_{n-m}-b_m b_{n-m})\cos\theta^- -(a_m b_{n-m}+a_{n-m}b_m)\sin\theta^-\right] \tag{25}$$

where the free index n runs from 1 to N with N being the number of frequency components retained in the solution. The coefficients are defined as

$$\alpha_1=\left(1+3pk_n^2h^2\right)C-2qk_nh^2\omega_n,\qquad \alpha_2=(3pk_nC-q\omega_n)h^2$$
$$\alpha_s=\left(C/4h+rCk_n^2h-sk_nh\omega_n\right)h_x+(3pk_nC-q\omega_n)h^2(k_n)_x \tag{26}$$

with

$$\delta^+=k_{n+m}-k_m-k_n,\qquad \theta^+=\int_0^x\delta^+dx$$
$$\delta^-=k_{n-m}+k_m-k_n,\qquad \theta^-=\int_0^x\delta^+dx \tag{27}$$
$$\alpha^+=\frac{k_{n+m}-k_m}{\alpha_1+\alpha_2\delta^+},\qquad \alpha^-=\frac{k_{n-m}+k_m}{\alpha_1+\alpha_2\delta^-}.$$

For a given radian frequency ω_n and water depth h the corresponding wave number is obtained from the dispersion relationship of the wave model by solving the cubic equation:

$$\left(pCh^2\right)k_n^3-\left(q\omega_nh^2\right)k_n^2+Ck_n-\omega_n=0. \tag{28}$$

The wave number gradient $(k_n)_x$ may be directly calculated from the depth gradient h_x by using Eq. (19). The linear shoaling characteristics of the original equation are exactly preserved in the above formulation; this can be easily seen by comparing α_s with Eq. (18) when the imaginary part, Eq. (28), is set to zero.

Eqs. (24) and (25) are solved for the unknown components $a_n(x)$ and $b_n(x)$ by employing the Runge–Kutta fourth-order formulation. Once the $a_n(x)$s and $b_n(x)$s are obtained the free surface may be constructed from $\eta(x,t)=\sum_{n=1}^{N}[a_n\cos(\omega_nt-\int k_ndx)+b_n\sin(\omega_nt-\int k_ndx)]$. Here, the integrated forms of the wave numbers are essential as they are spatially varying quantities. Evolution Eqs. (24) and (25) of wave model, Eq. (13), are now used for a sample simulation.

6.1. A numerical example: nonlinear wave evolutions over a submerged bar

Beji and Battjes (1994) made measurements of regular and random nonlinear waves passing over a submerged bar for numerical testing purposes. Briefly, the bathymetry is constant with 0.4 m water depth for the first 0.3 m, then an upslope of 1:20 follows for 6 m reducing the water depth to 0.1 m. For 2 m the depth is constant at 0.1 m then a downslope of 1:10 increases the water depth to 0.4 m in 3 m. Details of the experimental setup and conditions can be found in Beji and Battjes (1994). Here, the regular long wave case with $T=2$ s period and $H=0.02$ m wave height is simulated and compared with the experimental data. The test is a challenging one as it requires good nonlinear shoaling and dispersion characteristics, especially due to wave decomposition phenomenon taking place in the lee of the submerged bar. Classical Boussinesq equations could not cope with such a simulation as reported in Beji and Battjes (1994). Similarly, the original and regularized KdV equations could not be used for meaningful spectral solutions since no real wave numbers could be computed for the third and higher harmonic frequencies. The problem arises from large negative errors in the dispersion relations of these equations as previously indicated in Fig. 1. Necessarily then numerical simulations are performed only for two different β values, $\beta=0$ and $\beta=-1/20$, both corresponding to improved forms of the KdV equations derived in this work. Spectral solutions are carried out for $N=6$ harmonic frequencies, the basic frequency being set to the incident wave frequency. The harmonic components of the incoming wave at Station 1 of the experiments are introduced as boundary values at $x=0$ m. While the first harmonic or the primary wave, is quite well predicted the second harmonic is overestimated by both models as clearly seen in Fig. 3. For the first two harmonics virtually there are no differences between the predictions of the two models. For the third and fourth harmonics some differences are observed; despite its apparent inferiority in dispersion characteristics the consistent model $\beta=0$ seems to perform somewhat better than $\beta=-1/20$, especially for the third harmonic.

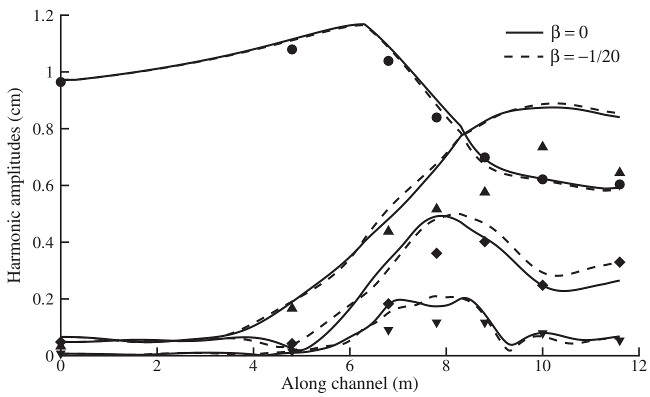

Fig. 3. Change in the first four harmonics over a submerged bar. Symbols are experimental data, lines are simulations of KdV equation.

7. Concluding remarks

A new form of the Korteweg & de Vries equation with improved linear dispersion and consistent linear shoaling characteristics is derived. The equation has improved linear dispersion characteristics in the sense that the phase celerity of the equation deviates less than 10% from the exact linear dispersion relationship for $kh = \pi$, which is virtually the deep water limit. The consistency in linear shoaling is achieved by ascertaining identical linear shoaling gradients as obtained from the wave equation and the energy flux concept. A challenging numerical simulation of nonlinear waves traveling over a submerged bar is performed using the new equation for two different β values, which correspond to the consistent model $\beta = 0$ and the dispersionally better but not consistent model $\beta = -1/20$. The comparisons with the experimental data reveal that the results obtained from two different parameters differ only slightly and the consistent model $\beta = 0$ may be preferred despite its apparently inferior dispersion properties. However; a different opinion may favor to improve the dispersion characteristics further instead of adhering to the consistency hence seek a best fit to the exact dispersion curve within a preset limit by adjusting the parameter β. Such an approach is equally arguable and may be tried for comparisons.

Finally, the improved unidirectional equation may readily be extended to include weakly two-dimensional effects as in a KP type equation. An extension in this direction with a robust and accurate finite-difference discretization is planned as a future work.

References

Abbott, M.B., 1974. Continuous flows, discontinuous flows and numerical analysis. J. Hydraul. Res. 12–4, 417–467.

Abbott, M.B., Damsgaard, A., Rodenhuis, G.S., 1973. System 21, “Jupiter” (a design system for two dimensional nearly horizontal flows). J. Hydraul. Res. 11–1, 1–28.

Abbott, M.B., Petersen, H.M., Skovgaard, O., 1978. On the numerical modelling of short waves in shallow water. J. Hydraul. Res. 16–3, 173–203.

Abbott, M.B., McCowan, A.D., Warren, I.R., 1984. Accuracy of short-wave numerical models. J. Hydraul. Eng. 110–10, 1287–1301.

Baker Jr., G.S., Graves-Morris, P., 1980. Padé Approximants. Cambridge University Press (745 pp.).

Beji, S., Battjes, J.A., 1994. Numerical simulation of nonlinear wave propagation over a bar. Coast. Eng. 23, 1–16.

Beji, S., Nadaoka, K., 1996. A formal derivation and numerical modelling of the improved Boussinesq equations for varying depth. Ocean Eng. 23 (8), 691–704.

Beji, S., Nadaoka, K., 1999. A spectral model for unidirectional nonlinear wave propagation over arbitrary depths. Coast. Eng. 36, 1–16.

Benjamin, T.B., Bona, J.L., Mahony, J.J., 1972. Model equations for long waves in nonlinear dispersive systems. Philos. Trans. R. Soc. Lond. A 272, 47–78.

Boussinesq, J.V., 1872. Theory of waves and surges which propagate the length of a horizontal rectangular canal, imparting to the fluid contained within the canal velocities that are sensibly the same from the top to the bottom. (Translated by A. C. J. Vastano and J. C. H. Mungall, March 1976.). J. Math. Pures Appl. 17, 55–108.

Bryant, P.J., 1973. Periodic waves in shallow water. J. Fluid Mech. 59, 625–644.

Freilich, M.H., Guza, R.T., 1984. Nonlinear effects on shoaling surface gravity waves. Philos. Trans. R. Soc. Lond. A 3, 1–41.

Kadomtsev, B.B., Petviashvili, V.I., 1970. On the stability of solitary waves in weakly dispersive media. Sov. Phys. Dokl. 15, 539–541.

Korteweg, D.J., de Vries, G., 1895. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philos. Mag. 39, 422–443.

Lamb, H., 1932. Hydrodynamics. Dover Publications, New York (738 pp.).

Madsen, P.A., Sørensen, O.R., 1992. A new form of the Boussinesq equations with improved linear dispersion characteristics: part 2. A slowly-varying bathymetry. Coast. Eng. 8, 183–204.

Madsen, P.A., Sørensen, O.R., 1993. Bound waves and triad interactions in shallow water. Ocean Eng. 20–4, 359–388.

Madsen, P.A., Murray, R., Sørensen, O.R., 1991. A new form of the Boussinesq equations with improved linear dispersion characteristics. Coast. Eng. 15, 371–388.

Mei, C.C., 1989. The Applied Dynamics of Ocean Surface Waves. World Scientific (740 pp.).

Mei, C.C., Le Méhauté, B., 1966. Note on the equations of long waves over an uneven bottom. J. Geophys. Res. 71, 393–400.

Miles, J.W., 1980. Solitary waves. Annu. Rev. Fluid Mech. 12, 11–43.

Miles, J.W., 1981. The Korteweg–de Vries equation: a historical essay. J. Fluid Mech. 106, 131–147.

Nwogu, O., 1993. Alternative form of Boussinesq equations for nearshore propagation. J. Waterw. Port Coast. Ocean Eng. 9 (6), 618–638.

Peregrine, D.H., 1966. Calculations on the development of an undular bore. J. Fluid Mech. 25–02, 321–330.

Peregrine, D.H., 1967. Long waves on a beach. J. Fluid Mech. 27–04, 815–827.

Rayleigh, Lord, 1876. On waves. Philos. Mag. 1, 257–279.

Simarro, G., 2013. Energy balance, wave shoaling and group celerity in Boussinesq-type wave propagation models. Ocean Model. 72, 74–79.

Witting, J.M., 1984. A unified model for the evolution of nonlinear water waves. J. Comput. Phys. 56, 203–236.