THE SECOND COASTAL ENGINEERING WORKSOP
10-11 OCTOBER 1994, YILDIZ-İSTANBUL

Spectral Domain Solution of An Improved Boussinesq Model

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Abstract

A spectral domain solution of an improved Boussinesq model is proposed for simulation of unidirectional nonlinear waves propagating over a gently varying depth. The wave propagation model accommodates improved dispersion characteristics as well as mild variations in bathymetry. The numerical technique is based on the Fourier series expansions of velocity and surface displacement with spatially varying coefficients. Performance of the scheme is tested against the laboratory measurements of nonlinear wave propagation over a bar and found to agree well with the experimental data.

1. Introduction

Recent years have witnessed a constantly increasing interest towards the modelling of nonlinear aspects of ocean waves, particularly in the coastal zone. The trend originated from the need to elucidate many observed phenomena which could not be accounted for otherwise. Wave skewness related sediment transport, influence of breaking on the surf-zone processes, and effects of harmonic generation on the characteristics of a wave field are but the most striking examples of such phenomena (Doering and Bowen, 1986, Nadaoka, *et al.*, 1989, Kojima, *et al.*, 1990, Beji and Battjes, 1993).

Although nonlinear wave propagation models are still in their infancy, at least in terms of practical applications, there are evidences that these models, when augmented with appropriate generation and dissipation mechanisms, may well be the prototypes of the standard models to come. At present, weakly nonlinear-weakly dispersive wave models, namely the Boussinesq-like models, seem to be the most promising ones for practical near-shore zone applications. These depth-integrated equations at once reduce a three-dimensional problem to the solution of an equivalent (within the approximations made) two-dimensional problem. Besides this reduction in dimension, these models pose less computational problems in comparison with the fully nonlinear, fully dispersive models.

Basically there are three different ways of tackling with a Boussinesq-type model. The most common one is to employ the finite-differences formulations to approximate the derivatives (Peregrine, 1967). The second approach is to make use of Fourier series expansions with slowly varying coefficients (Bryant, 1973), and finally the third is to adopt the method of characteristics (Mei and LeMéhauté, 1966). The last approach was to some extend used in the past but, due to the inconvenience of irregular grid locations, is no longer preferable. Here, we shall concentrate on the second approach only for a one-dimensional model, and compare the performance of the scheme against the experimental measurements over a bar (Beji and Battjes, 1993).

2. Governing Equations

The classical Boussinesq equations suffer from the inherent disadvantage of being shallow water equations. To extend their applicable range numerous attempts have been made (Witting, 1984, Madsen, *et. al.*, 1991, Nwogu, 1993). Here we shall use one such an improved model due to Beji and Nadaoka (1994), which might be viewed as a rectified version of Madsen and Sørensen's (1992) work. Accordingly, the dispersion characteristics of the present model are improved to the extent that waves with wavelengths equal to depth may be represented with acceptable errors in amplitude and celerity.

The improved Boussinesq equations, as formulated by Beji and Nadaoka (1994) are

$$\eta_t+\nabla.[(h+\eta)q]=0, \tag{1}$$

$$q_t+(q.\nabla)q+g\nabla\eta=(1+\beta)\frac{h}{2}\nabla[\nabla.(hq_t)]+\beta\frac{gh}{2}\nabla[\nabla.(h\nabla\eta)] \\ -(1+\beta)\frac{h^2}{6}\nabla(\nabla.q_t)-\beta\frac{gh^2}{6}\nabla(\nabla.\nabla\eta), \tag{2}$$

where q=(u,v) is the two-dimensional depth-averaged velocity vector, η is the surface displacement, h=h(x,y) is the varying water depth as measured from the still water level, g is the gravitational acceleration, and β is a constant. The subscript t stands for partial differentiation with respect to time and ∇ for the horizontal gradient operator. The z-axis is taken vertically upwards with the origin at the undisturbed free surface.

In linearized forms equations (1) and (2) lead to the following dispersion relation

$$\frac{c^2}{gh}=\frac{(1+\beta k^2h^2/3)}{[1+(1+\beta)k^2h^2/3]}, \tag{3}$$

where c=|c| is the phase speed, $k^2=k_x^2+k_y^2$ and k_x, k_y are the components of wave-number vector **k** in the x- and y-directions respectively.

Matching equation (3) with a first-order Padé expansion of the linear theory dispersion relation requires $\beta=0$, which in turn reduces equation (3) to the classical Boussinesq dispersion relation. On the other hand, an expansion correct to the second-order dictates $\beta=1/5$, and naturally results in better agreement with the exact form of the linear theory dispersion relation. Here, $\beta=1/5$ is used in all computations.

For unidirectional waves equations (1) and (2) reduce to

$$\eta_t+[(h+\eta)u]_x=0, \tag{4}$$

$$u_t+\frac{1}{2}(u^2)_x+g\eta=(1+\beta)hh_xu_{xt}+(1+\beta)\frac{h^2}{3}u_{xxt}+\beta ghh_x\eta_{xx}+\beta g\frac{h^2}{3}\eta_{xxx}, \tag{5}$$

where the subscripts denote partial differentiation with respect to the indicated variable. Note for $\beta=0$ equations (4) and (5) reduce to the set derived by Peregrine (1967).

3. Series Expansion Solution

A possible way of solving weakly nonlinear differential equations, such as Boussinesq or KdV equations, is to use a Fourier series expansion with "slowly varying" coefficients for the variable concerned (Whitham, 1974). The term "slowly varying" simply implies that variations in coefficients over the characteristic length or time scale are small so that derivatives of the coefficients higher than the first may be neglected. Such an approach eventually leads to an infinite number of first order coupled nonlinear differential equations, which are truncated at an appropriate number and solved via a suitable integration technique.

To date there are three spectral Boussinesq models that account for wave shoaling; namely, consistent and dispersive shoaling models of Freilich and Guza (1984), and the recent work of Madsen and Sørensen (1993). In principle all these models make the same assumptions and follow the same procedure for obtaining the evolution equations corresponding to the governing equations selected. In such an approach, existence of linear shoaling terms forces one to make further sacrifices in solution accuracy. This point is not explicit and may only be seen when the formulation of a model for horizontal bottom is compared with that of a shoaling model. The model developed here is free of such a drawback and therefore may well compete with horizontal-bottom models, which are inherently more accurate than any shoaling model given so far.

An unorthodox approach is now introduced to solve the improved Boussinesq model in frequency domain. We begin by re-casting (4) and manipulating (5) as

$$hu_x=-[\eta_t+h_xu+(\eta u)_x]\,, \tag{6}$$

$$g\eta_x+\frac{(1+\beta)}{3}h\eta_{xtt}-\beta ghh_x\eta_{xx}-\frac{\beta}{3}gh^2\eta_{xxx}=-[u_t+\frac{1}{2}(u^2)_x+\frac{(1+\beta)}{3}h_x\eta_{tt}]\,, \tag{7}$$

where use has been made of the linearized continuity equation in re-expressing the dispersive and shoaling terms in the momentum equation. Linearization in this manipulation is justified because in non-dimensional form both terms, u_{xxt} and u_{xt}, are proportional $\mu^2(=k^2h^2$, k and h are respectively a typical wave-number and depth), and the contribution due to the inclusion of the nonlinear terms would be on the order of $\varepsilon\mu^2$ ($\varepsilon=a/h$, a is a typical wave amplitude) which is even smaller and an order higher than the classical Boussinesq models extend. Note that no zeroth order term has been replaced and that the linear dispersion relation corresponding to (6) and (7) is still the same as (3).

By keeping two variables, η and u, we have diverged from the customary way of dealing with such problems and assumed the burden of solving for an extra variable, namely u. It is expected that this extra effort will pay back in terms of solution accuracy. Moreover, the availability of the velocity field is always an asset in practical applications.

Let the surface displacement and vertically averaged velocity be expressed as

$$\begin{aligned}\eta(x,t) &= \sum_{n=-\infty}^{+\infty}\alpha_n(x)\,e^{i(\omega_n t-k_n x)}\\ u(x,t) &= \sum_{n=-\infty}^{+\infty}\beta_n(x)\,e^{i(\omega_n t-k_n x)}\end{aligned} \tag{8}$$

in which $\alpha_n(x)$'s and $\beta_n(x)$'s are slowly varying complex coefficients dependent only upon x, ω_n's are cyclic frequencies satisfying $\omega_n=n\omega_0$ with ω_0 being the lowest wave frequency present, k_n's are wave numbers as determined from (3), and $i=\sqrt{-1}$. It is assumed that the following equalities hold as well:

$$\alpha_{-n}=\alpha^*_n,\quad \beta_{-n}=\beta^*_n,\quad \omega_{-n}=-\omega_n,\quad k_{-n}=-k_n \tag{9}$$

where asterisk denotes complex conjugate.

There may be questions as to the generality of the series expansions in (8). First of all, for free wave components travelling at their own phase speeds, it is obvious that the expressions are proper. As for the so-called "bound" waves, which are phase locked to a carrier or primary wave, the necessary phase adjustments at every spatial point are taken care of by the individual components of the complex coefficients as long as the final evolution equations correspond to the solution of a non-dispersive representation. This point is of crucial importance for obtaining correct phases for bound waves, and accomplished via a simple change of variables, as was done by Bryant (1973).

The algebra involved in derivation of the evolution equations is straightforward but lengthy and shall not be repeated here. The procedure is as follows. First, the series expansions are substituted into the governing equations. The second and third spatial derivatives of $\alpha_n(x)$'s arising from the term $\beta g h^2 \eta_{xxx}/3$ are all neglected as well as the products of derivatives. The dispersion relation, (3), is then used to simplify some of the expressions containing ω_n and k_n. After introducing the new variables, $A_n=\alpha_n \exp(-ik_n x)$ and $B_n=\beta_n \exp(-ik_n x)$, comes the final step of substituting $\frac{1}{2}(a_n-ib_n)$ for A_n and $\frac{1}{2}(c_n-id_n)$ for B_n, which in turn yield the following evolution equations for the spatially varying coefficients $a_n(x)$, $b_n(x)$, $c_n(x)$, and $d_n(x)$.

$$
\begin{aligned}
\rho_0 (a_n)_x &= \rho_r a_n - \rho_i b_n - \omega_n d_n \\
&\quad - \frac{1}{2}\sum_{m=1}^{N-n} \gamma_{n,m}(c_m d_{n+m} - d_m c_{n+m}) + \frac{1}{4}\sum_{m=1}^{n-1} \delta_{n,m}(c_m d_{n-m} + d_m c_{n-m}) \\
\rho_0 (b_n)_x &= \rho_i a_n + \rho_r b_n + \omega_n c_n \\
&\quad + \frac{1}{2}\sum_{m=1}^{N-n} \gamma_{n,m}(c_m c_{n+m} + d_m d_{n+m}) - \frac{1}{4}\sum_{m=1}^{n-1} \delta_{n,m}(c_m c_{n-m} - d_m d_{n-m}) \\
h (c_n)_x &= -h_x c_n - \omega_n b_n - \frac{1}{2}\sum_{m=1}^{N-n} \gamma_{n,m}(a_m d_{n+m} - d_m a_{n+m} + c_m b_{n+m} - b_m c_{n+m}) \\
&\quad + \frac{1}{2}\sum_{m=1}^{n-1} \delta_{n,m}(a_m d_{n-m} + b_m c_{n-m}) \\
h (d_n)_x &= -h_x d_n + \omega_n a_n + \frac{1}{2}\sum_{m=1}^{N-n} \gamma_{n,m}(a_m c_{n+m} + c_m a_{n+m} + b_m d_{n+m} + d_m b_{n+m}) \\
&\quad - \frac{1}{2}\sum_{m=1}^{n-1} \delta_{n,m}(a_m c_{n-m} - b_m d_{n-m})
\end{aligned}
\tag{10}
$$

with the coefficients given as

$$\rho_0 = \frac{\omega_n^2}{k_n^2 h} + \frac{2}{3}\beta g k_n^2 h^2 \qquad \rho_r = [\frac{(1+\beta)}{3}\omega_n^2 - \beta g k_n^2 h]\, h_x \qquad \rho_i = \frac{2}{3}\beta g h k_n^3 h^2$$

$$\gamma_{n,m} = k_m - k_{n+m} \qquad \delta_{n,m} = k_m + k_{n-m} \tag{11}$$

where N is the number of harmonics present in the computation, and n=1, 2, 3, ... , N. From the symmetry, all the summations running from m=1 to n-1 may be arranged to run from m=1 to (n-1)/2 when n is odd, to n/2 when n is even (see Mei, 1989, p.603).

For a selected number of harmonic components, N, the above equations yield 4N number of coupled first order nonlinear differential equations which must be solved simultaneously. There are two well-known integration techniques suitable for this type differential equations: Runge-Kutta method, and Bulirsch-Stoer method. Here, because of its superior qualities, the latter is preferred and adopted in its original form (Bulirsch and Stoer, 1966). A crucial point is the agreement between the boundary values of the surface displacement and those of the velocity. For each frequency component, the averaged-velocity corresponding to a given surface displacement must be specified in accordance with the linearized continuity equation, $u_0=c_0\eta_0/h_0$, where c_0 is the phase celerity computed from (3) for the water depth h_0, and η_0 is the incident wave amplitude. Any other specification of the velocity causes superfluous oscillations in the harmonic components, reflecting the disagreement between what is imposed at the boundary and what is required by the evolution equations.

Once the Fourier coefficients are obtained at discrete spatial points they are substituted into the non-dispersive versions of (8) to obtain the time variations of the surface elevation and vertically averaged horizontal velocity.

4. Comparisons with Experimental Measurements

Beji and Battjes (1993) carried out laboratory measurements of nonlinear wave transformations over a submerged bar. Due to harmonic generation in the shoaling region and subsequently their release behind the bar in the deepening part of the flume, the case itself is a rather severe test for any nonlinear wave propagation model. The entire program of measurements is rather involved and the reader is directed to this particular reference. The cross-section of the wave flume and the locations of the measurement stations are given in Figure 1.

Here, we shall compare only two cases for regular waves. In the first case the incident wave period is T=2 seconds and the incident wave amplitude is 1 cm at a depth of 40 cm. The measured wave profile at station 1 (as indicated in Figure 1) is given as input. Figure 2 compares the measured (solid lines) and computed (dots) wave profiles for stations 2-7. The agreement is virtually perfect except for station 7, where the free higher-harmonics become quite manifest. Since these higher-harmonics have wavelengths much shorter in comparison with the incident wave, they may be considered as deep water waves. Thus, the small discrepancies observed at statio 7 is attributed to the fact that at this particular station the applicable range of the wave model is exceeded.

The second case is actually quite beyond the applicable range of standard Boussinesq models. Indeed, even for an improved Boussinesq model this test poses an extreme case. Here, the incident wave period is T=1.25 seconds and the incident wave amplitude is 1.3 cm at 40 cm depth. As observed from Figure 3 the numerical model is not doing as well as the previous case but still producing acceptable results. Since the incident wave is comparatively short, the resonance conditions are not satisfied hence the harmonic generation is quite low. For this reason alone, at station 7 the free higher-harmonics are much less than the previous case.

5. Concluding Remarks

A spectral domain solution of the improved Boussinesq equations is developed for unidirectional wave propagation over gently varying depths. The improved model makes it possible to propagate waves with wavelengths equal to depth and accounts for shoaling effects. The numerical approach appears to be quite accurate as comparisons with the severe experimental cases show. It is hoped that future tests of this model will be made with actual field measurements for breaking waves, after a revision of the scheme to take into account the dissipative role of breaking.

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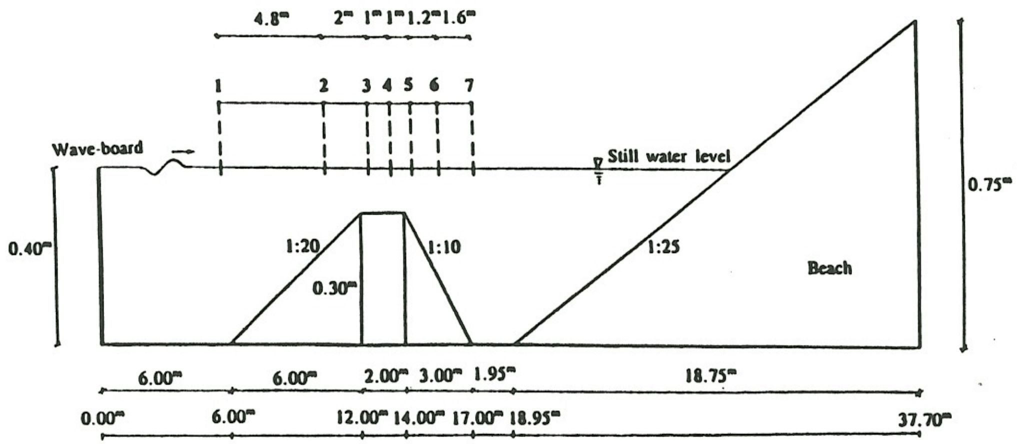
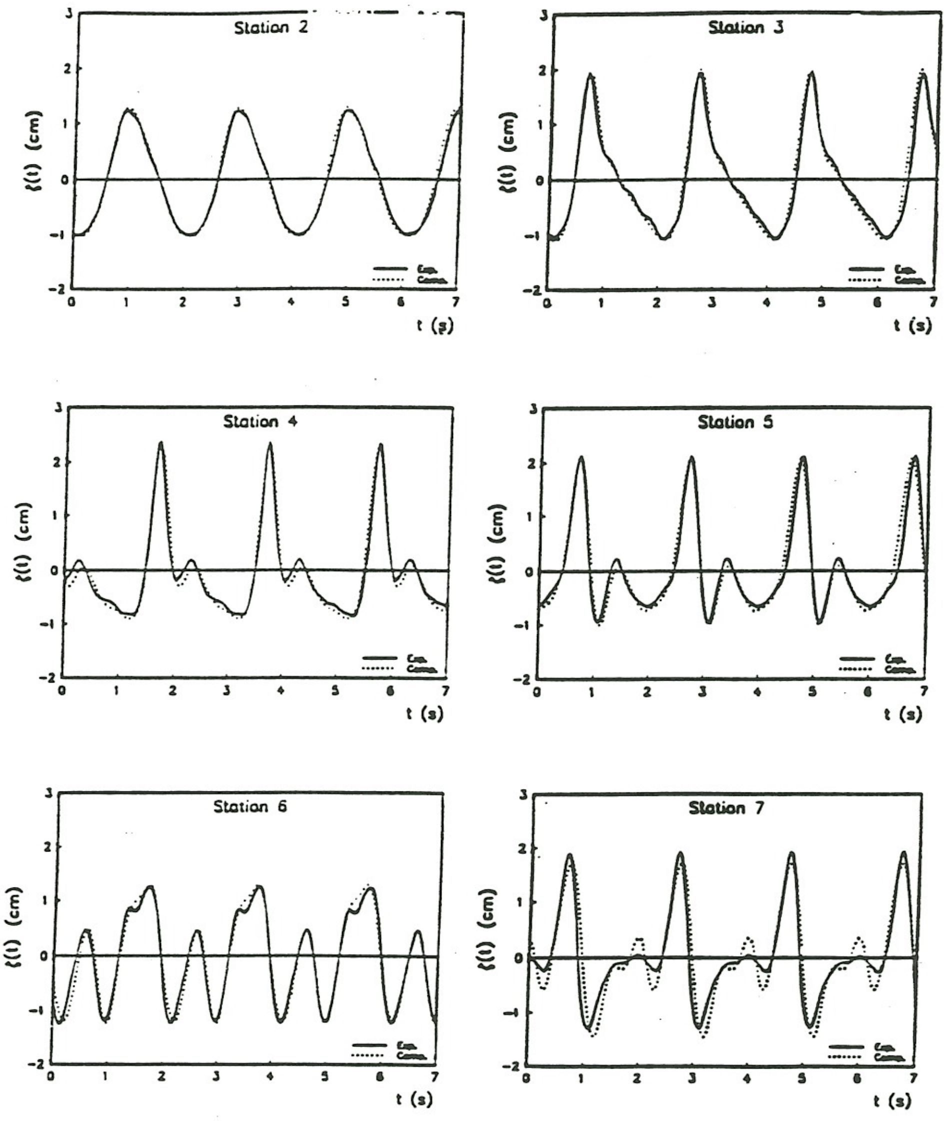
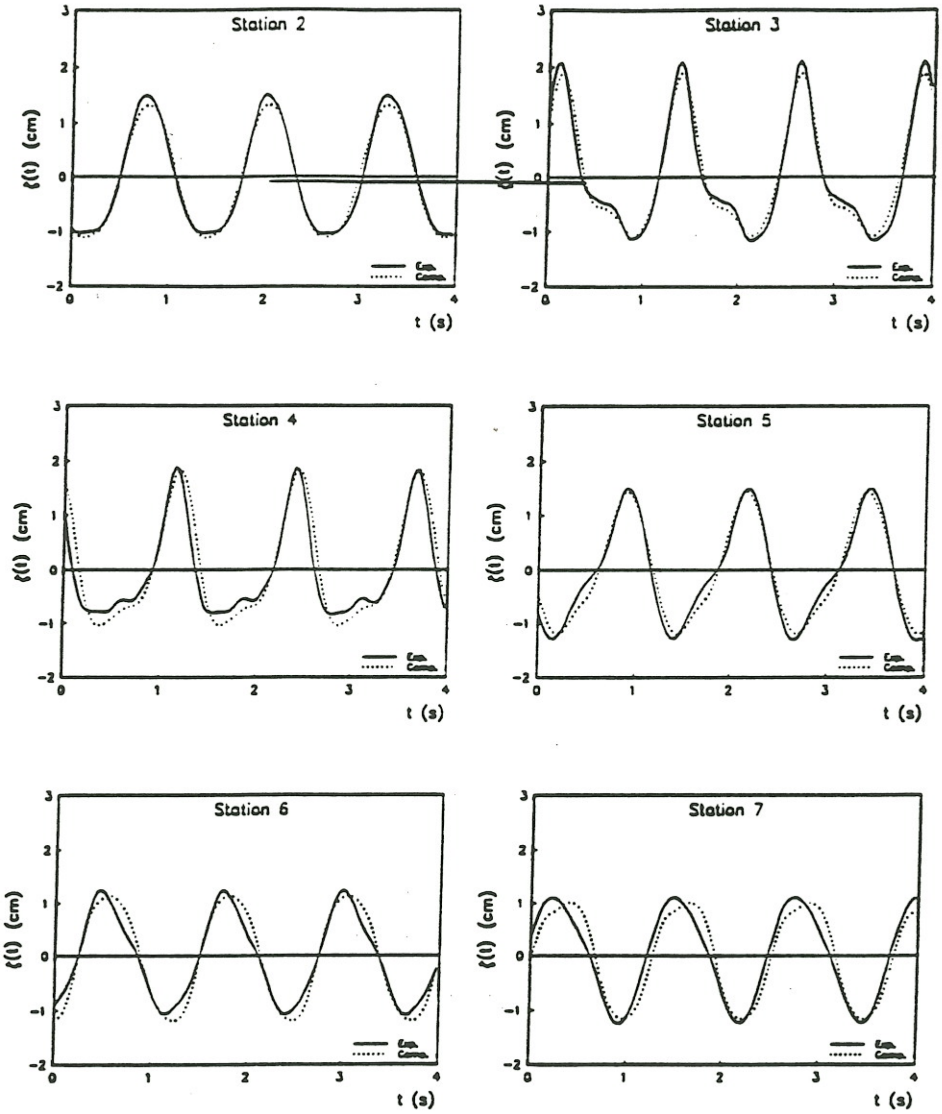

4.8m
2m
1m
1m
1.2m
1.6m
1
2
3
4
5
6
7
Wave-board
Still water level
0.40m
0.75m
1:20
1:10
0.30m
1:25
Beach
6.00m
6.00m
2.00m
3.00m
1.95m
18.75m
0.00m
6.00m
12.00m
14.00m
17.00m
18.95m
37.70m

Figure 1



Frequency Domain Solution
Comparison of measurement and computation for regular long waves

Figure 2



Frequency Domain Solution
Comparison of measurement and computation for regular short waves

Figure 3