Spectral Modelling of Nonlinear Wave Shoaling and Breaking over Arbitrary Depths

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Abstract

A spectral model is formulated for the simulation of breaking and non-breaking unidirectional waves propagating over arbitrary depths. The evolution equations corresponding to the conservative wave model of Beji and Nadoka (1997a) are given first and then an empirical generalization of Battjes's (1986) energy dissipation model for breaking waves is introduced. The generalized dissipation terms are embedded into the conservative evolution equations so as to make the spectral model capable of simulating wave height decay due to breaking. The evolution equations with the breaking terms are then used to reproduce Horikawa and Kuo's (1966) and Beji and Battjes's (1993) laboratory measurements of breaking waves as well as the field measurements of Nakamura and Katoh (1992) in the surf zone. Despite the crude approximations involved in the formulation of the breaking dissipation, the comparisons show rather good agreements.

Introduction

With the rapid advance of computational facilities, recent years have seen an increasing interest towards the simulation of the nonlinear aspects of the surface waves, particularly in the nearshore region where these effects are observed to be most appreciable. Wave skewness related sediment transport, effects of harmonic generation on the characteristics of a wave field, and influence of breaking on the surf-zone processes are the most striking examples of such phenomena (Doering and Bowen, 1986; Freilich and Guza, 1984; Nadaoka *et al.*, 1989). For practical applications the nonlinear wave models are not yet in common use, however, there are evidences that when augmented with appropriate generation and dissipation mechanisms, these models may well be the prototypes of the commercial models to come.

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In this work a spectral model useable both for breaking and nonbreaking unidirectional waves is developed by employing the recently introduced wave equation of Beji and Nadaoka (1997a), which is a combined unidirectional form of the fully dispersive weakly nonlinear wave equations of Nadaoka *et al.* (1994, 1997), in conjunction with Battjes's (1986) dissipation model. The resulting evolution equations are numerically solved for various test cases to demonstrate the capabilities of the model. Comparisons with the measurements reveal quite acceptable agreements even for considerably complicated cases.

Wave Model

Quite recently, Beji and Nadaoka (1997a) have introduced the following wave equation for weakly-nonlinear, narrow-banded, unidirectional waves travelling over arbitrary depths

$$\begin{aligned} C_g\eta_t &+ \frac{1}{2}C_p(C_p+C_g)\eta_x - \frac{(C_p-C_g)}{k^2}\eta_{xxt} - \frac{C_p(C_p-C_g)}{2k^2}\eta_{xxx} \\ &+ \frac{1}{2}\left[C_p(C_g)_x + (C_p-C_g)(C_p)_x\right]\eta + \frac{3}{4}g\left(3 - 2\frac{C_g}{C_p} - \frac{k^2C_p^4}{g^2}\right)\left(\eta^2\right)_x = 0, \end{aligned} \tag{1}$$

where C_p, C_g, and k are respectively the phase and group velocities and the wave-number computed according to the linear theory dispersion relation for a dominant wave frequency ω and a given local depth h, the subscripts x and t indicate partial differentiation with respect to space and time, respectively.

If the relative depth is small $C_p = C_g = (gh)^{1/2}$ and equation (1) reduces to the combined unidirectional form of Airy's nonlinear non-dispersive equations. Allowing the lowest-order dispersion by letting $C_p = (gh)^{1/2}(1 - k^2h^2/6)$ and $C_g = (gh)^{1/2}(1 - k^2h^2/2)$ leads to the KdV equation. For infinitely deep water waves the model equation admits the second-order Stokes waves as solution. Thus, the model equation may be viewed as a unified nonlinear wave model describing evolution of a narrow-banded wave field from infinitely deep to very shallow waters with smooth transition.

An important aspect in modelling wave transformations over variable sea bed is the linear shoaling characteristics of the wave model employed. If the incident wave frequency coincides with the prescribed wave frequency of equation (1) the linear shoaling is predicted exactly for any relative depth. This point may be easily demonstrated using the approach introduced by Madsen and Sørensen (1992). An incident wave of the from $\eta = a(x)\exp[\omega t - \int k(x)dx]$ is substituted into the wave equation and the higher derivatives of $a(x)$ and $k(x)$ are neglected so that an expression for the spatial variation of $a(x)$ is obtained. Carrying out this procedure for equation (1) results in $a_x/a = -(C_g)_x/2C_g$, which is exactly the expression obtained from the constancy of energy flux.

Evolution Equations

Using the wave equation (1) and expressing the surface elevation as a Fourier series with spatially varying amplitudes and phases, Beji and Nadaoka (1997b) derived a set of evolution equations describing the spatial changes of the component wave amplitudes of a prescribed incident wave field:

$$\begin{aligned}\frac{da_n}{dx} &= -\frac{\alpha_s}{\alpha_1}a_n \\ &+\beta\sum_{m=1}^{N-n}\alpha^-[(a_m b_{n+m} - a_{n+m}b_m)\cos\theta^- + (a_m a_{n+m} + b_m b_{n+m})\sin\theta^-] \\ &+\frac{1}{2}\beta\sum_{m=1}^{n-1}\alpha^+[(a_m b_{n-m} + a_{n-m}b_m)\cos\theta^+ + (a_m a_{n-m} - b_m b_{n-m})\sin\theta^+]\end{aligned} \tag{2a}$$

$$\begin{aligned}\frac{db_n}{dx} &= -\frac{\alpha_s}{\alpha_1}b_n \\ &-\beta\sum_{m=1}^{N-n}\alpha^-[(a_m a_{n+m} + b_m b_{n+m})\cos\theta^- - (a_m b_{n+m} - a_{n+m}b_m)\sin\theta^-] \\ &-\frac{1}{2}\beta\sum_{m=1}^{n-1}\alpha^+[(a_m a_{n-m} - b_m b_{n-m})\cos\theta^+ - (a_m b_{n-m} + a_{n-m}b_m)\sin\theta^+]\end{aligned} \tag{2b}$$

where

$$\begin{aligned}\alpha_1 &= \frac{1}{2}C_p(C_p + C_g) - 2\frac{(C_p - C_g)}{k^2}\omega_n k_n + \frac{3C_p(C_p - C_g)}{2k^2}k_n^2, \\ \alpha_2 &= \frac{(C_p - C_g)}{k^2}\left(\frac{3}{2}C_p k_n - \omega_n\right), \\ \alpha_s &= \frac{1}{2}[C_p(C_g)_x + (C_p - C_g)(C_p)_x] + \frac{(C_p - C_g)}{k^2}\left(\frac{3}{2}C_p k_n - \omega_n\right)(k_n)_x, \\ \beta &= \frac{3}{4}g\left(3 - 2\frac{C_g}{C_p} - \frac{k^2 C_p^4}{g^2}\right), \quad \delta^+ = k_{n-m} + k_m - k_n, \quad \delta^- = k_{n+m} - k_m - k_n, \\ \alpha^+ &= \frac{k_{n-m} + k_m}{\alpha_1 + \alpha_2\delta^+}, \quad \alpha^- = \frac{k_{n+m} - k_m}{\alpha_1 + \alpha_2\delta^-}, \qquad \theta^+ = \int_0^x \delta^+ dx, \quad \theta^- = \int_0^x \delta^- dx,\end{aligned} \tag{3}$$

in which k_n for each component is computed from the linear dispersion relation of the wave model for the local depth $h(x)$ and the radian frequency $\omega_n = n\Delta\omega$, $\Delta\omega$ being the frequency of resolution:

$$C_g\omega_n - \frac{1}{2}C_p(C_p + C_g)k_n + \frac{(C_p - C_g)}{k^2}\omega_n k_n^2 - \frac{C_p(C_p - C_g)}{2k^2}k_n^3 = 0, \tag{4}$$

and $(k_n)_x$ appearing in α_s is obtained from (4) by differentiating it with respect to x.

The free index n runs from 1 to N, resulting in $2N$ number of nonlinearly coupled first-order differential equations for the unknown components $a_n(x)$ and $b_n(x)$. Once the $a_n(x)$s and $b_n(x)$s are obtained the free surface elevation may be constructed from $\eta = \Sigma a_n \cos\phi_n + b_n \sin\phi_n$, with $\phi_n = \omega_n t - \int k_n dx$. Various numerical integration techniques (*e.g.*, Adams-Bashford-Moulton, Bulirsch-Stoer, Runge-Kutta) are available for the integration of equation (2). Here, the Runge-Kutta fourth-order formulation is preferred as it proved to be the fastest while being as reliable as the other more sophisticated integration methods.

Energy Dissipation Due to Breaking: A Generalization of Battjes's model

An empirical approach is now adopted to account for the dissipative role of wave breaking so as to render the spectral model operational even for breaking waves. To this end we first refer to Battjes's(1986) dissipation model for periodic waves:

$$\frac{\partial P}{\partial x} + D = 0 \qquad \text{with} \qquad D = \frac{B}{4\gamma^3}\frac{\rho g H^2}{T}\left(\frac{H}{h}\right)^4, \tag{5}$$

where ρ is the water density, g the gravitational acceleration, H the wave height, T the wave period, $P = EC_g$ the energy flux with $E = \frac{1}{8}\rho g H^2$ and $C_g = (gh)^{1/2}$ (shallow water). $B = O(1)$ is a calibration coefficient independent of any variable, and finally γ is the breaking coefficient which is *inversely* proportional to the height of the foam region in a breaker. The latter meaning of γ is especially emphasized for an *ad hoc* modification which is taken up later.

We have selected the above dissipation model of periodic waves rather than the stochastic description of Battjes and Janssen (1978) because our evolution equations are essentially based on a deterministic wave-by-wave analysis that traces the spatial changes of simple sinusoidal components with definite periods. From this simplified (the effects of phase-locking ignored) point of view it at once becomes evident that a dissipation model for periodic waves is the most appropriate choice.

Equation (5) is applicable only to shallow water waves and it is desirable to extend it to deep water so that it will be in accord with equation (1), which is valid for arbitrary depths. It is fully understood that such an extension is artificial and not in line with the physical considerations followed in derivation of (5); therefore, it is best to view this extension as purely empirical.

First of all the ratio H/h in (5) is viewed as the *wave steepness* for shallow water waves, the most important non-dimensional variable dictating the dissipation rate. A generalized wave steepness for waves over arbitrary depths has been introduced as gH/C_p^2 (Beji, 1995) with C_p denoting the wave phase speed according to linear theory. Thus, we propose the following modified form of (5) as the dissipation model

$$\frac{H_x}{H} = -\frac{B}{2\pi\gamma^3}\frac{\omega}{C_g}\left(\frac{gH}{C_p^2}\right)^4, \tag{6}$$

in which the linear shoaling term implicitly present in (5) has been removed since the wave model itself includes the linear shoaling properly, and C_g in its full form as defined by linear theory is adopted instead of the shallow water approximation. Note that for shallow waters $C_p^2 = gh$ and the ratio gH/C_p^2 reduces to H/h, which is identical with the original expression. On the other hand, for deep waters gH/C_p^2 tends to kH and, unlike H/h, still gives a non-zero dissipation due to breaking so long as the deep water wave steepness kH is appreciable.

If the wave is represented as $a(x)\cos\phi + b(x)\sin\phi$, ϕ being the phase angle, then, according to (6), the spatial rates of reduction of the amplitude components due to breaking are

$$\frac{a_x}{a} = -\frac{B}{2\pi\gamma^3}\frac{\omega}{C_g}\left(\frac{2g\sqrt{a^2+b^2}}{C_p^2}\right)^4, \qquad \frac{b_x}{b} = -\frac{B}{2\pi\gamma^3}\frac{\omega}{C_g}\left(\frac{2g\sqrt{a^2+b^2}}{C_p^2}\right)^4, \tag{7}$$

respectively for $a(x)$ and $b(x)$. The above formulation is valid only for simple sinusoidal wave forms; the evolution equations contain not only a single component but a number of components with different frequencies. It is therefore necessary to introduce some approximations to use the formulation (7) in (2). A plausible approach is to assume that each harmonic component is dissipated according to (7) while the main dissipation mechanism gH/C_p^2 comprises the total effect of all the components. In other words, *all* the wave components are dissipated in proportion to the *resultant* wave height, should it becomes large enough to induce breaking. Thus, for the wave breaking dissipation portions of (2a) and (2b) one can write

$$\left(\frac{da_n}{dx}\right)_d = -\frac{B}{2\pi\gamma^3}\frac{\omega_n}{(C_g)_n}\left(\frac{gH}{C_p^2}\right)^4 a_n, \qquad \left(\frac{db_n}{dx}\right)_d = -\frac{B}{2\pi\gamma^3}\frac{\omega_n}{(C_g)_n}\left(\frac{gH}{C_p^2}\right)^4 b_n, \tag{8}$$

where the resultant wave height is approximated simply as $H \simeq 2\sqrt{\Sigma(a_n^2+b_n^2)}$ while C_p is, as before, the local phase celerity computed for the dominant wave frequency ω and local depth h. $(C_g)_n$ denotes the group velocity corresponding to the frequency ω_n and local depth h according to linear theory.

Earlier it has been indicated that the parameter γ may be viewed as a measure of the extend of the foaming region in a breaker (*i.e.*, the smaller the γ the stronger the breaking). Battjes (1986) uses $\gamma = 0.6$ for constant depth and $\gamma = 0.7 + 5S$ (S is the bottom slope) for varying depth. In general there is no strict rule for the selection of γ (see Battjes and Stive, 1985). Here, we shall do a final modification and introduce the following definition of γ for use in our computations.

$$\gamma = 0.6\left[1+\left(\frac{\mu}{1+\mu}\right)\gamma^*\right]+\left(\frac{1}{1+\mu}\right)5S, \tag{9}$$

where $\mu = kh$ is the relative depth for the dominant wave, and γ^* a constant compensating for the relatively smaller dissipation rate of a deep water breaking wave by making γ larger. Typically, γ^* may be selected in the range of $0 \leq \gamma^* \leq 1$, and we use $\gamma^* = 0.3$ throughout.

Note that for very shallow water waves μ tends to zero and equation (9) becomes identical with Battjes's choice $\gamma = 0.7+5S$, except for the constant 0.7. The selection of 0.6 instead of 0.7 as a constant was done merely for including the constant depth value as a special case. For deep water waves (9) becomes

$\gamma = 0.6(1 + \gamma^*)$, which is independent of bottom slope and larger than 0.6, as deemed by the physical nature of white capping in contrast to depth-induced breaking.

After introducing such extensive modifications to Battjes's original formulation it becomes unavoidable to redefine the constant B. Battjes typically uses $B = 2$; after a few trials we have selected $B = 1$. In what follows we use $B = 1$, and γ as defined in (9) with $\gamma^* = 0.3$ for *all* the computational results presented. No other calibration is introduced despite the different nature of the cases considered. Furthermore, various tests with different γ^* values and different functional forms of γ have revealed no appreciable sensitivity and the overall dissipation model has been found to be quite robust.

Thus, after incorporating the dissipative effect of wave breaking by simply adding the expressions in (8) to the right-hand sides of (2a) and (2b) the final evolution equations may simulate breaking waves. It is worthwhile to indicate that the evolution equations with the breaking terms may be used for nonbreaking waves as well since the waves steepness term gH/C_p^2, which acts as a breaking criteria, becomes small for small amplitudes thus rendering the dissipation terms inactive.

Sample Simulations

The evolution equations formulated above are now used for sample simulations of breaking waves. Nonbreaking waves are covered in Beji and Nadaoka (1997b) and will not be considered here.

(a) Periodic waves breaking on constant depth

For a simple but fundamental demonstration we compare the results of the numerical solution of the present wave model with the exact analytical solution of Battjes's original model for an experimental case of Horikawa and Kuo (1966) for breaking waves over a horizontal bottom. Battjes shows that his analytical solution agrees well with the measurements of Horikawa and Kuo (1986).

Figure 1 shows the comparison for $h = 0.1$ m depth and $H_b/h = 0.8$ for the linearized equations, that is, when $N = 1$ in the evolution equations. The agreement with Battjes's original model (hence with Horikawa and Kuo's experimental data) is good and supports the validity of the approximations made in deriving (8).

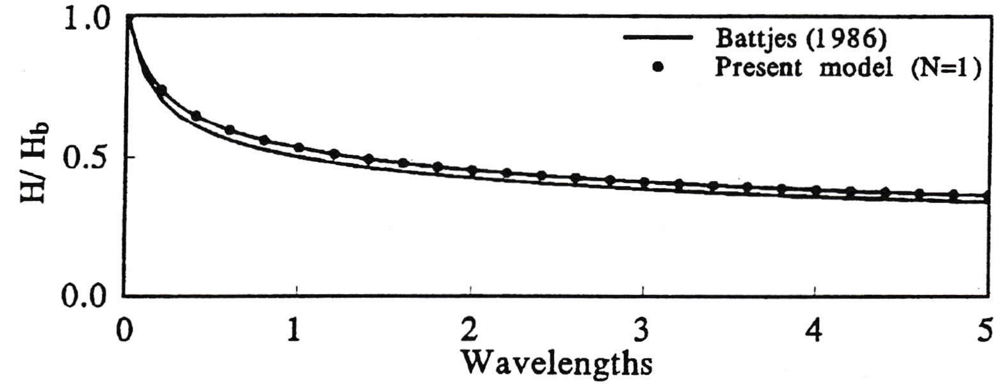

Figure 1: Wave height decay due to breaking over a horizontal bottom (Horikawa and Kuo, 1966). $h = 0.1$ m and $H_b/h = 0.8$, H_b is the initial breaking waveheight.

(b) Random waves breaking over a submerged bar

Beji and Battjes (1993) performed laboratory experiments for investigating the nonlinear transformations of random breaking waves travelling over a submerged bar. Their experimental measurements for random waves initially having a JONSWAP type wave spectrum with peak frequencies of $f_p = 0.4$ Hz (long waves) and $f_p = 1.0$ Hz (short waves) are simulated here.

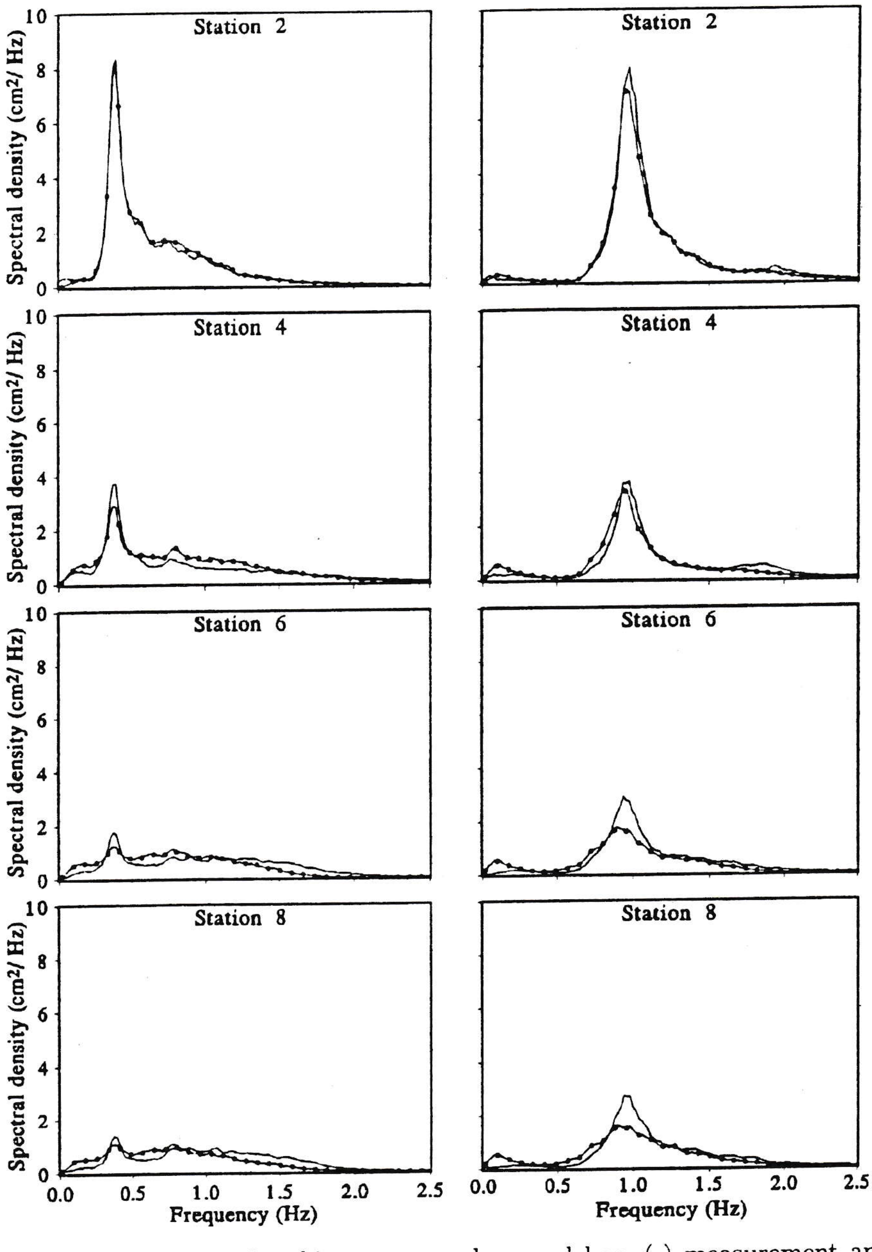

Figure 2: Random waves breaking over a submerged bar, (-) measurement and (•) computation. Left column: long waves, right column: short waves.

The initial wave amplitudes were such that most of the breaking waves were *plunging* type. (*Spilling* type breakers were also simulated successfully but the comparisons could not be included here due to space limitations.) For long waves the dominant frequency for the model equation was set to the mean frequency of the incident spectrum while for short waves the peak frequency was used since the frequency width of the spectrum was comparatively narrower. The records of the surface elevation at Station 1 were divided into 80 segments of 512 data points and then each segment was Fourier transformed. Out of the 256 unique pairs the first 150 Fourier components, which covered a frequency range of 0.0195-3.0 Hz, were found to be quite sufficient to represent the incident wave spectrum hence the spectral model was run for 80 different realizations with $\Delta\omega = 2\pi \times 0.0195$ rd/s and $N = 150$, using the measured Fourier components as the incoming boundary condition at Station 1.

Figure 2 compares the measured and computed spectra at four selected stations for the long (left column) and short (right column) wave cases, respectively. The spectra were obtained after ensemble averaging all the realizations and frequency smoothing three neighbouring components. Each spectrum then has 480 degrees of freedom and 6.5 % normalized standard error. The agreement of the computations with the both sets of measurements is remarkably good, especially if the complicated nature of the wave transformations is considered.

(c) Breaking waves in surf zone

Nakamura and Katoh (1992) carried out field measurements from 25 February 1989 to 1 March 1989 at the Hazaki Oceanographical Research Facility (HORF) near Kashima, Japan. The site of the field observations is a natural sandy beach facing the Pacific Ocean. Ten ultrasonic wave gauges were used, of which seven were installed on the 427 m-long observatory pier while the remaining three were deployed at distances of 1.3, 2.1, and 3.2 km from the shoreline. Data was continuously sampled at a rate of 2 Hz for two-hour durations, at six-hour time intervals. A quantitative assessment made by Nwogu *et al.* (1992) using the maximum entropy method provides convincing evidence that for practical purposes the wave field in this particular region may be considered unidirectional.

Since the first seven measurement stations were located in the surf zone the majority of the waves were breaking and these stations were most suitable for testing the wave model. Thus, only the Stations 7, 6, 5, 4, and 3 were considered; the measured data at Station 7 served as the incoming boundary condition. Comparisons were made for the data of February 28, which was recorded in the aftermath of a severe storm and represented a rather rough sea state.

The collected data at Station 7 was segmented into 24 groups of 512 data points and Fourier transformed. Of the 256 unique transformed pairs, the first 128 components which covered a frequency range between 0.0039 to 0.5 Hz were considered sufficient to represent the incident spectrum. The dominant frequency of the wave model was set to the peak frequency of the incident wave spectrum and the computations were performed for 24 different realizations with $\Delta\omega = 2\pi \times 0.0039$ rd/s and $N = 128$. Figure 3 shows the measured and computed spectra at Stations 7, 6, 5, 4, and 3. Each spectrum has 192 degrees of freedom and 10% normalized standard error. The agreement is quite reasonable, especially if allowances are made for the uncertainties involved in the unidirectionalty of waves and the exact form of the bottom topography.

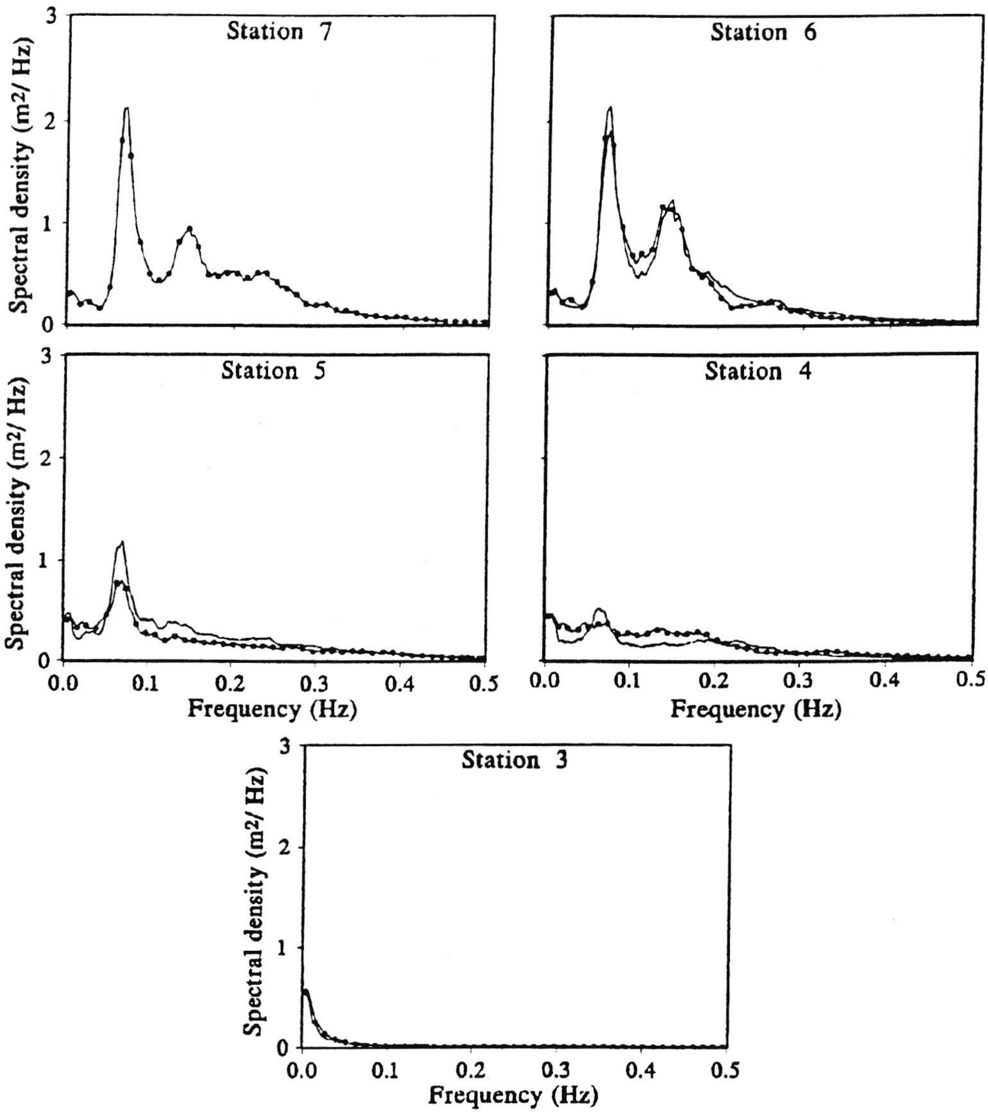

Figure 3: Breaking waves in the surf zone (Nakamura and Katoh, 1992), (-) measurement and (•) computation.

Concluding Remarks

A spectral model which is employable for both breaking and nonbreaking waves over arbitrary depths has been developed by modifying the spectral domain formulation of a weakly-nonlinear unidirectional wave equation through a generalized version of Battjes's (1986) dissipation model. The performance of the spectral model for breaking waves has been tested for several different cases and found to be acceptable. Quite importantly, the generalized dissipation mechanism has proved to be insensitive to the calibration coefficients; a single initial calibration was enough for diversely different simulations.

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