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A Fundamental Relationship of Polynomials and Its Proof

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Abstract

A fundamental algebraic relationship for a general polynomial of degree n is given and proven by mathematical induction. The stated relationship is based on the well-known property of polynomials that the n^{th}-differences of the subsequent values of an n^{th}-order polynomial are constant.

Keywords

Polynomials of Degree n, n^{th}-Order Finite-Differences, Recurrence Relationship for Polynomials

1. Introduction

The "Fundamental Theorem of Algebra" states that a polynomial of degree n has n roots. Its first assertion in a different form is attributed to Peter Rothe in 1606 and later Albert Girard in 1629. Euler gave a clear statement of the theorem in a letter to Gauss in 1742 and at different times Gauss gave four different proofs (see [1], p. 292-306).

A nearly as important property of a polynomial is the constancy of the n^{th}-differences of its subsequent values. To clarify this point let us begin with some demonstrations. While it is customary to use polynomials with real coefficients, here a second-order polynomial with complex coefficients is considered first,

$$P_2(x) = (1+i)x^2 - 3ix + 2 \tag{1}$$

where $i = \sqrt{-1}$ is the imaginary unit. Taking a real starting point $x_0 = -2$ and a real step value $s = 1$ the following Table 1 of differences can be established for the subsequent values of the polynomial.

The first differences are computed by taking the differences of the subsequent values of the polynomial as in $P_2(-2) - P_2(-1) = (6+10i) - (3+4i) = 3+6i$.

Table 1. Second-order differences for a sample second-order polynomial.

m	0	1	2	3	4
$x_0 + ms$	-2	-1	0	1	2
$P_2(x_0 + ms)$	6+10i	3+4i	2+0i	3-2i	6-2i
First differences	3+6i	1+4i	-1+2i	-3+0i	
Second differences	2+2i	2+2i	2+2i		

The second differences are obtained similarly using the first difference values: $(3+6i)-(1+4i)=2+2i$.

Expressing the first differences in terms of polynomial values $P_2(-2)-P_2(-1)=3+6i$ and $P_2(-1)-P_2(0)=1+4i$, the first value of the second differences may be written as

$$\begin{aligned}&\left[P_2(-2)-P_2(-1)\right]-\left[P_2(-1)-P_2(0)\right]\\&=P_2(-2)-2P_2(-1)+P_2(0)=2+2i\end{aligned} \tag{2}$$

which is a particular form, $n=2$, of the general theorem presented in Section 2. The constant value $2+2i$ of the second-differences can be calculated from the general expression $(-1)^n n!a_0 s^n$ where n is the degree of the polynomial and a_0 the coefficient of the n^{th}-order term. For this particular example $n=2$ and $a_0=1+i$ hence the constant becomes $(-1)^2 2!(1+i)1^2=2+2i$ as found above.

Another example is now given for a third-order polynomial with real coefficients

$$P_3(x)=2x^3-x^2-3x+5 \tag{3}$$

In this example a complex starting point $x_0=1-i$ and a complex step value $s=-3+2i$ are used so that **Table 2** of differences is constructed, where the constant value of the third-differences can be calculated from the general formula $(-1)^n n!a_0 s^n$ as $(-1)^3 3!2(-3+2i)^3=-108-552i$.

A direct connection with the finite-difference approximation of derivatives of a polynomial is of course possible. Finite-difference approximation of the third-derivative of a third-order polynomial is given by

$$P_3'''(x)=\frac{P_3(x_0+3s)-3P_3(x_0+2s)+3P_3(x_0+s)-P_3(x_0)}{s^3} \tag{4}$$

where s is the incremental step. If the third-order polynomial is defined as $P_3(x)=a_0x^3+a_1x^2+a_2x+a_3$ its third-derivative is $P_3'''(x)=6a_0$. Now using this in (4) results in

$$P_3(x_0)-3P_3(x_0+s)+3P_3(x_0+2s)-P_3(x_0+3s)=-6a_0s^3 \tag{5}$$

which exactly corresponds to the tabulated constant of third-differences. A remarkable point is that while finite-difference approximations are typically formulated for real and relatively small incremental step sizes, for the general expression no such restrictions apply: the incremental step s may be complex and arbitrarily large while the result is always exact.

Table 2. Third-order differences for a sample third-order polynomial.

m	0	1	2	3	4	5
$x+ms$	1-i	-2+i	-5+3i	-8+5i	-11+7i	-14+9i
$P_3(x+ms)$	-2+i	4+23i	24+417i	166+1735i	538+4529i	1248+9351i
First differences	-6-22i	-20-394i	-142-1318i	-372-2794i	-710-4822i	
Second differences	-14-372i	-122-924i	-230-1476i	-338-2028i		
Third differences	-108-552i	-108-552i	-108-552i			

Finally, a possible application of (5) or its general form for an n^{th}-order polynomial, is its use as a recurrence formula for evaluating a given polynomial at equal intervals once the polynomial is evaluated at n distinct points. For instance for a third-order polynomial it is sufficient to know $P_3(x_0)$, $P_3(x_0+s)$, and $P_3(x_0+2s)$ to obtain $P_3(x_0+3s)$ from (5). Then, by setting x_0 to x_0+s in (5), $P_3(x_0+4s)$ can be obtained from the same recurrence relationship and continuing in this manner gives $P_3(x_0+5s)$, $P_3(x_0+6s)$, etc. with considerably less arithmetic operations compared to straightforward evaluation of polynomial.

2. Main Theorem and Proof

The main theorem which expresses the constancy of n^{th}-order differences for an n^{th}-order polynomial is stated first and then proven by the method of induction.

Theorem 1

For an n^{th}-order polynomial $P_n(x)=a_0x^n+a_1x^{n-1}\cdots+a_{n-1}x+a_n$ with $a_0\neq 0$ the following relationship holds

$$\sum_{m=0}^{n}(-1)^m\binom{n}{m}P_n(x+ms)=(-1)^n n!a_0s^n \tag{6}$$

where $n\geq 1$ and a_j's, x, $s\in\mathbb{R}$ or $\mathbb{C}$.

Proof of Theorem 1. The base case: Setting $n=1$ in (6) results in

$$\binom{1}{0}P_1(x)-\binom{1}{1}P_1(x+s)=(-1)^1 1!a_0s^1 \tag{7}$$

Substituting $P_1(x)=a_0x+a_1$ and $P_1(x+s)=a_0(x+s)+a_1$ gives

$$(a_0x+a_1)-\left[a_0(x+s)+a_1\right]=-a_0s \tag{8}$$

which is correct.

The inductive step: Assuming that the statement (6) holds true for any integer n it is now proven that it also holds true for $(n+1)$:

$$\sum_{m=0}^{n+1}(-1)^m\binom{n+1}{m}P_{n+1}(x+ms)=(-1)^{n+1}(n+1)!a_0s^{n+1} \tag{9}$$

$P_{n+1}(\bar{x})$ can be expressed in terms of $P_n(\bar{x})$ as

$$P_{n+1}(\bar{x})=\bar{x}P_n(\bar{x})+a_{n+1} \tag{10}$$

Letting $\bar{x} = x + ms$ in (10) and using it in (9) result in

$$\sum_{m=0}^{n+1} (-1)^m \binom{n+1}{m} \left[(x+ms) P_n (x+ms) + a_{n+1} \right] = (-1)^{n+1} (n+1)! a_0 s^{n+1} \tag{11}$$

Since $\sum_{m=0}^{n+1} (-1)^m \binom{n+1}{m} = 0$ for any n (odd or even) the summation proportional to the constant a_{n+1} vanishes, reducing (11) to

$$x \sum_{m=0}^{n+1} (-1)^m \binom{n+1}{m} P_n (x+ms) + s \sum_{m=0}^{n+1} (-1)^m m \binom{n+1}{m} P_n (x+ms) = (-1)^{n+1} (n+1)! a_0 s^{n+1} \tag{12}$$

Making use of $\sum_{m=0}^{n+1} \binom{n+1}{m} = \sum_{m=0}^{n} \binom{n}{m} + \sum_{m=1}^{n+1} \binom{n}{m-1}$ ([2], p. 882) in the first summation above results in

$$x \left[\sum_{m=0}^{n} (-1)^m \binom{n}{m} P_n (x+ms) + \sum_{m=1}^{n+1} (-1)^m \binom{n}{m-1} P_n (x+ms) \right] + s \sum_{m=0}^{n+1} (-1)^m m \binom{n+1}{m} P_n (x+ms) = (-1)^{n+1} (n+1)! a_0 s^{n+1} \tag{13}$$

By re-defining the running index in the second summation (13) becomes

$$x \left[\sum_{m=0}^{n} (-1)^m \binom{n}{m} P_n (x+ms) + \sum_{m=0}^{n} (-1)^{m+1} \binom{n}{m} P_n \left[x + (m+1)s \right] \right] + s \sum_{m=0}^{n+1} (-1)^m m \binom{n+1}{m} P_n (x+ms) = (-1)^{n+1} (n+1)! a_0 s^{n+1} \tag{14}$$

Since x may be assigned to any value, substituting $x + s$ in place of x in the base case (6) reveals that $\sum_{m=0}^{n} (-1)^m \binom{n}{m} P_n \left[x + (m+1)s \right]$ is too equal to the same quantity: $(-1)^n n! a_0 s^n$. Noting in the second summation in (14) that $(-1)^{m+1} = -(-1)^m$ renders the terms in square brackets zero. Thus, to complete the proof the remaining equality in the second line of (14) must be proven:

$$s \sum_{m=0}^{n+1} (-1)^m m \binom{n+1}{m} P_n (x+ms) = (-1)^{n+1} (n+1)! a_0 s^{n+1} \tag{15}$$

For $m = 0$ the first term of the summation in (15) is zero hence bringing no contribution. Therefore, we can start the summation from $m = 1$ without any error. Then, the summation may be expressed as

$$\sum_{m=1}^{n+1} (-1)^m m \binom{n+1}{m} = \sum_{m=1}^{n+1} (-1)^m m \frac{(n+1)!}{(n+1-m)! m!} = \sum_{m=1}^{n+1} (-1)^m \frac{(n+1)!}{(n+1-m)! (m-1)!}$$

$$
\begin{aligned}
&= \sum_{m=1}^{n+1} (-1)^m (n+1) \frac{n!}{(n+1-m)!(m-1)!} \\
&= \sum_{p=0}^{n} (-1)^{p+1} (n+1) \frac{n!}{(n-p)!\,p!} \\
&= (n+1) \sum_{p=0}^{n} (-1)^{p+1} \binom{n}{p}
\end{aligned}
$$

where an obvious change of running index $m = p + 1$ has been implemented in the final stage. Employing the last expression obtained above after changing p to m for the summation of (15) results in

$$
s(n+1) \sum_{m=0}^{n} (-1)^{m+1} \binom{n}{m} P_n\left[x + (m+1)s\right] = (-1)^{n+1} (n+1)!\, a_0 s^{n+1} \tag{16}
$$

As indicated above, the main theorem may also be stated as $\sum_{m=0}^{n} (-1)^m \binom{n}{m} P_n\left[x + (m+1)s\right] = (-1)^n\, n!\, a_0 s^n$. Using this in (16) yields

$$
s(n+1)(-1)(-1)^n\, n!\, a_0 s^n = (-1)^{n+1} (n+1)!\, a_0 s^{n+1} \tag{17}
$$

which proves that the proposition holds true for $(n + 1)$ as well. ☐

References

[1] Smith, D.E. (1959) A Source Book in Mathematics. Dover Publications, New York.

[2] Abramowitz, M. and Stegun, I.A. (1972) Handbook of Mathematical Functions. Dover Publications, New York.