

Wave and Current Forces on Objects in Water

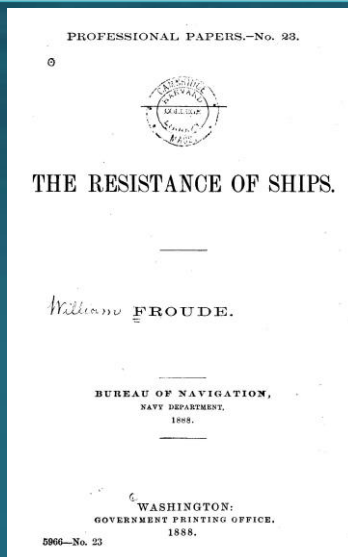
Prof. Dr. Serdar Beji
Istanbul Technical University

Landmark contributions

Froude & Froude

First comprehensive treatment of ship resistance

1888



THE FORCE EXERTED BY SURFACE WAVES ON PILES

J. R. MORISON, M. P. O'BRIEN, J. W. JOHNSON AND S. A. SCHAAF, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA.

THE FORCE EXERTED BY SURFACE WAVES ON PILES

The force exerted by unbroken surface waves on a cylindrical object, such as a pile, which extends from the bottom upward above the wave crest, is made up of two components, namely:

1. A drag force proportional to the square of the velocity which may be represented by a drag coefficient having substantially the same value as for steady flow, and
2. A virtual mass force proportional to the horizontal component of the accelerative force exerted on the mass of water displaced by the pile.

These relationships follow directly from wave theory and have been confirmed by measurements in the Fluid Mechanics Laboratory of the University of California, Berkeley. The maximum force exerted by breakers or incipient breakers is impulsive in nature, reaching a value much greater than that produced by unbroken waves but enduring for only a short time interval. This impulsive force represents the ultimate development of the accelerative force and is produced by the steep wave front and large horizontal acceleration at the front of a breaker. This impulsive force greatly exceeds the drag force computed from the particle velocities of the breaker.

The reader is cautioned that these preliminary results are applicable only to single piles without leaning and are likely to be modified somewhat where multiple piles are driven, one within the influence of the other or where multiple piles are connected by submerged bracing. This paper is essentially a preliminary report submitted at this time because of the current importance of wave forces in the design of offshore structures. An extended series of additional experiments is planned for the near future.

Theoretical Relationships

For the sake of simplicity of treatment, the theory will be developed from the equations for waves of small amplitude. The horizontal displacement of a water particle is described by the equation

$$X = \frac{H}{2} \frac{\cosh \frac{2\pi}{L}(d+z)}{\sinh \frac{2\pi}{L}d} \sin \frac{2\pi t}{T} \quad (1)$$

The $\sin \theta$ term (italicized) is negative for $0 < \theta < \pi$ and positive for $\pi < \theta < 2\pi$. Values of A and A' appear in Fig. 1.

The horizontal component of the orbital velocity is obtained by differentiation with respect to time as

$$u = -\frac{\pi H}{T} \frac{\cosh \frac{2\pi}{L}(d+z)}{\sinh \frac{2\pi}{L}d} \cos \frac{2\pi t}{T} \quad (2)$$

The acceleration of the water particles at any position is obtained by again differentiating with respect to time as

$$\frac{\partial u}{\partial t} = \frac{\pi^2 H}{T^2} \frac{\cosh \frac{2\pi}{L}(d+z)}{\sinh \frac{2\pi}{L}d} \sin \frac{2\pi t}{T} \quad (3)$$

where

H = wave height—ft
 L = wave length—ft
 d = still water depth—ft
 z = depth below the still water measured negatively downward—ft
 T = wave period—sec
 t = time—sec

The force exerted on a differential section, ds , in length is

$$dF = \left[C_D \left(\frac{\pi D}{4} \right) \frac{\partial u}{\partial t} + C_M \frac{\partial^2 u}{\partial t^2} \right] ds \quad (4)$$

where

D = pile diameter—ft
 ρ = water mass density—slugs/ft³
 C_D = coefficient of mass
 C_M = coefficient of drag

Substituting for $\partial u/\partial t$ and $\partial^2 u/\partial t^2$ from Eqs. (3) and (2), the force per unit length at any position z is

$$\frac{dF}{ds} = \frac{\pi^2 D H^2}{4 T^2} \left[-C_M \sin \theta + C_D \cos^2 \theta \right] \quad (5)$$

The $\sin \theta$ term (italicized) is negative for $0 < \theta < \pi$ and positive for $\pi < \theta < 2\pi$. Values of A and A' appear in Fig. 1.

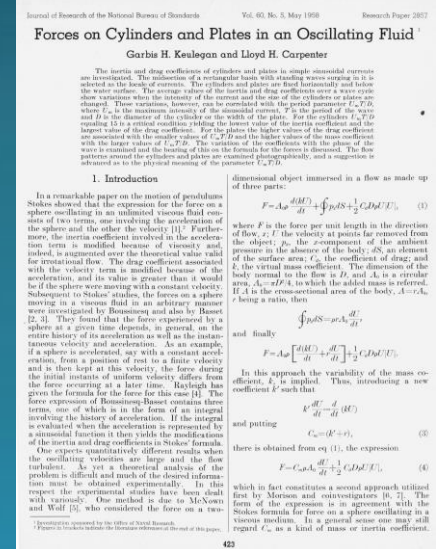
1950

Morison, O'Brien, Johnson, and Schaaf
Forces exerted by surface waves on piles

Keulegan & Carpenter

Forces on cylinders and plates in an oscillating fluid

1958



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$$\frac{\partial u}{\partial t} = \frac{-2\pi^2 H}{T^2} \frac{\cosh \frac{2\pi}{L}(d+z)}{\sinh \frac{2\pi d}{L}} \sin \frac{2\pi t}{T} \quad \dots (3)$$

where

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The force exerted on a differential section, dz, in length is

$$dF = \left[C_M \left(\rho \frac{\pi D^2}{4} \right) \frac{\partial u}{\partial t} \pm C_D \frac{\rho D}{2} u^2 \right] dz \quad \dots (4)$$

where

D = pile diameter — ft

ρ = water mass density — slugs/ft³

C_M = coefficient of mass

C_D = coefficient of drag

Substituting for $\partial u/\partial t$ and u^2 from Eqs. (3) and (2), the force per unit length at any position z is

$$\frac{dF}{dz} = \frac{\pi^3 \rho D H^2}{2T^2} \left[-f_M \sin \theta \pm f_D \cos^2 \theta \right] \quad \dots (5)$$

$$\frac{f_M}{C_M} = \pi A \frac{D}{H}; \quad \frac{f_D}{C_D} = A^2; \quad \theta = \frac{2\pi t}{T}$$

$$A = \frac{\cosh \frac{2\pi}{L}(d+z)}{\sinh \frac{2\pi d}{L}} \cdot \begin{cases} +\cos^2 \theta & \text{for } 0 < \theta < \pi/2 \\ \frac{3\pi}{2} < \theta < 2\pi \\ -\cos^2 \theta & \text{for } \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases}$$

The $\sin \theta$ term (inertia) is negative for $0 < \theta < \pi$ and positive for $\pi < \theta < 2\pi$. Values of A and A² appear in Fig. 1

Morison Equation

Forces exerted by surface waves on piles

Morison, O'Brien, Johnson, and Schaaf (1950)

Total force = Inertia force + Drag force

$$dF = \rho C_M \frac{\partial U}{\partial t} d\mathcal{V} + \frac{1}{2} \rho C_D |U| U d\mathcal{A}_p$$

$$d\mathcal{V} = \frac{\pi}{4} D^2 dz$$

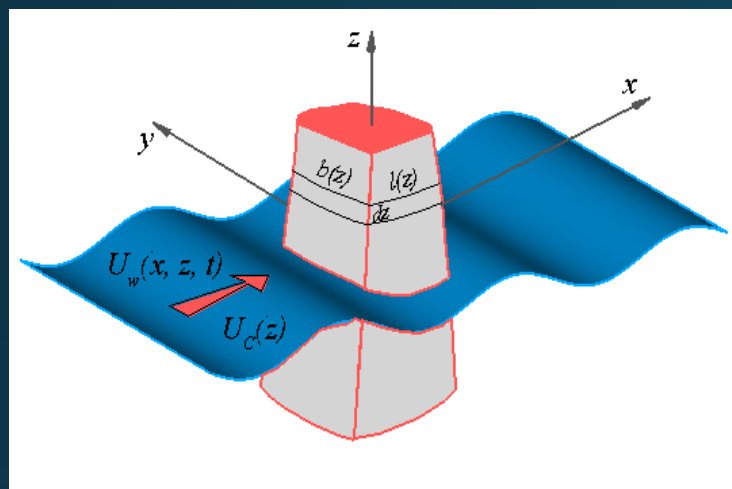
Displaced volume

$$d\mathcal{A}_p = D dz$$

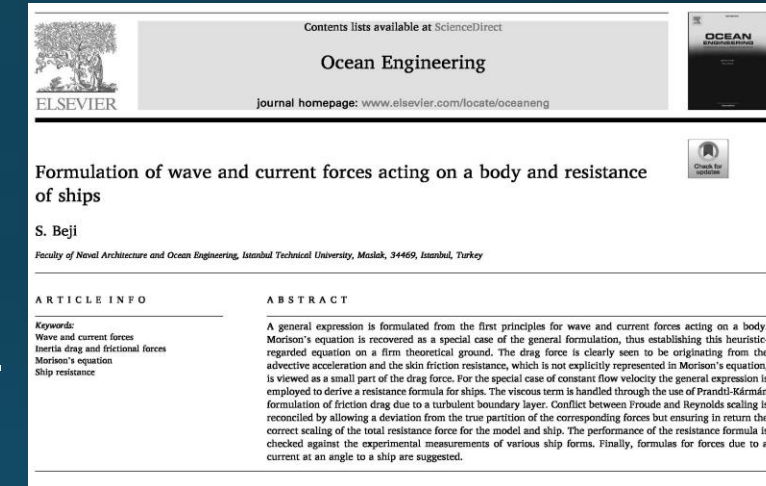
Projected frontal area

Momentum Equation for 1-D flow

Formulation of wave and current forces acting on a body and resistance of ships, Beji (2020)



$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}$$



Multiply by volume element $dV = dx dy dz$ and integrate over the object in the x – and y – directions

$$dF = \rho C_M \frac{\partial u}{\partial t} dV(z) + \frac{1}{2} \rho C_D u^2 dA_p(z) + \frac{1}{2} \rho C_F u^2 C(z) dz$$

$$C_F = 0.072 Re^{-1/5}$$

Friction coefficient (Prandtl – Kármán)

$$C(z)$$

Object perimeter

Morison Equation from momentum formulation

Waves only $u = U_w$ for a circular cylinder $d\mathcal{V}(z) = \frac{\pi}{4}D^2(z)dz$ $d\mathcal{A}_p = D(z)dz$ $\mathcal{C}(z) = \pi D(z)$

$$dF = \underbrace{\rho C_M \frac{\partial U_w}{\partial t} \frac{\pi}{4} D^2(z) dz}_{\text{Inertia force}} + \underbrace{\frac{1}{2} \rho C_D U_w |U_w| D(z) dz}_{\text{Drag force}} + \underbrace{\frac{1}{2} \rho C_F U_w |U_w| \pi D(z) dz}_{\text{Frictional force}}$$

Wave and current together $u = U_c + U_w$ for a circular cylinder of varying diameter

$$\begin{aligned} dF &= \rho C_M \frac{\partial U_w}{\partial t} \frac{\pi}{4} D^2(z) dz + \frac{1}{2} \rho C_D (U_c^2 + 2U_c U_w + U_w |U_w|) D(z) dz \\ &+ \frac{1}{2} \rho C_F (U_c^2 + 2U_c U_w + U_w |U_w|) \pi D(z) dz \end{aligned}$$

Ship resistance: General formulation

$$dF = \rho C_M \frac{\partial u}{\partial t} d\mathcal{V}(z) + \frac{1}{2} \rho C_D u^2 d\mathcal{A}_p(z) + \frac{1}{2} \rho C_F u^2 \mathcal{C}(z) dz$$

For a constant current speed $u = U$ vertically integrated form of this general force expression

$$R = \frac{1}{2} \rho C_D \mathcal{A}_p U^2 + \frac{1}{2} \rho C_F \mathcal{S}_w U^2$$

Inertial resistance Frictional resistance

\mathcal{A}_p : Projection area, \mathcal{S}_w : Wetted surface area, $C_F = 0.072 \alpha_f Re^{-1/5}$, $Re = UL/\nu$

$$R = \frac{1}{2} \rho C_D \mathcal{A}_p U^2 + \frac{1}{2} \rho \alpha_f \frac{0.072}{\sqrt[5]{L/\nu}} \mathcal{S}_w U^{1.8}$$

R. E. Froude's frictional resistance formula:

$$R_f = f(L) \mathcal{S}_w U^{1.825}$$

Ship resistance: Scaling

$$F_s = \frac{1}{2} \rho C_D \mathcal{A}_{ps} U_s^2 + \frac{1}{2} \rho \alpha_f \frac{0.072}{\sqrt[5]{U_s L_s / \nu}} \mathcal{S}_{ws} U_s^2$$

Froude scaling $U_s / \sqrt{g L_s} = U_m / \sqrt{g L_m}$. If $L_s = \lambda L_m$ then $U_s = \lambda^{1/2} U_m$, $\mathcal{A}_{ps} = \lambda^2 \mathcal{A}_{pm}$, etc.

$$F_s = \frac{1}{2} \rho C_D \lambda^3 \mathcal{A}_{pm} U_m^2 + \frac{1}{2} \rho \alpha_f \lambda^3 \frac{0.072}{\sqrt[5]{\lambda^{3/2} U_m L_m / \nu}} \mathcal{S}_{wm} U_m^2$$

$$\frac{F_s}{\lambda^3} = F_m = \frac{1}{2} \rho C_D \mathcal{A}_{pm} U_m^2 + \frac{1}{2} \rho \alpha_f \frac{0.072}{\sqrt[5]{\lambda^{3/2} U_m L_m / \nu}} \mathcal{S}_{wm} U_m^2$$

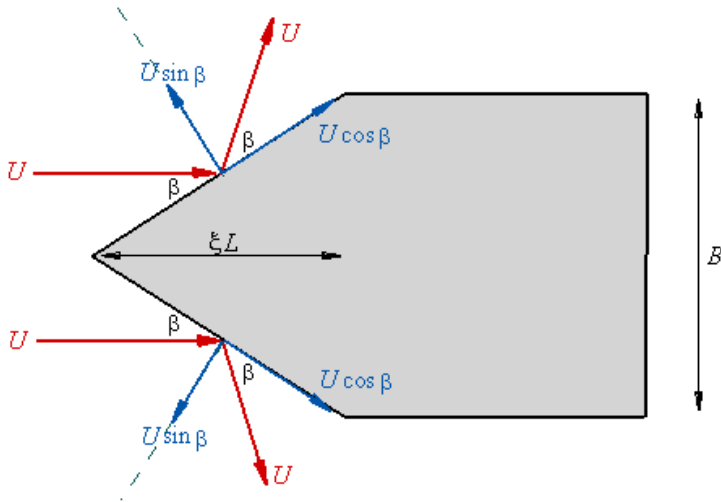
$\lambda = L_s / L_m = B_s / B_m = (\nabla_s / \nabla_m)^{1/3}$, ∇ : Displacement volume. Let $\nabla_s = \nabla_{ref} = 1 \text{ m}^3$ then $\lambda = (1 / \nabla_m)^{1/3}$,
 $\lambda^{3/2} = \nabla_m^{-1/2}$

$$F = \frac{1}{2} \rho C_D \mathcal{A}_p U^2 + \frac{1}{2} \rho \alpha_f \frac{0.072}{(Re / \sqrt{\nabla})^{1/5}} \mathcal{S}_w U^2$$

Ship resistance: Shape factor ξ and surface rise ζ

$$R = \frac{1}{2} \rho C_D \mathcal{A}_p U^2 + \frac{1}{2} \rho \alpha_f \frac{0.072}{(Re/\sqrt{\nabla})^{1/5}} \mathcal{S}_w U^2$$

A speculative argument in line with Newton's momentum approach



Velocity normal to surface $U \sin \beta$

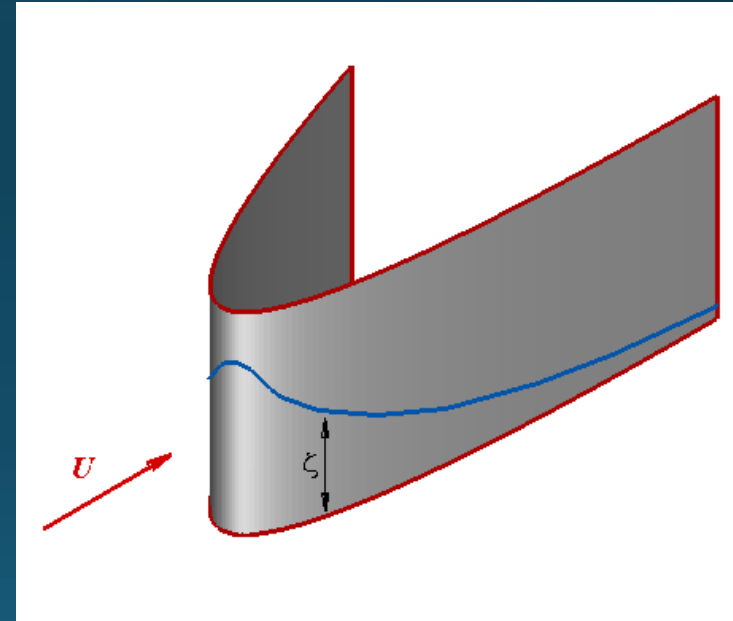
$$\frac{1}{2} \rho \alpha_d \mathcal{A}_p (U \sin \beta)^2$$

$$\sin^2 \beta = \frac{1}{1 + (2\xi L/B)^2}$$

In general $\xi \leq 1/2$

Free surface level increase ζ

$$\frac{1}{2} \rho (U \sin \beta)^2 + \rho g z_r = 0 + \rho g (z_r + \zeta)$$



$\alpha_d = 1$ and $\alpha_f = 1$, ξ : Only tuning parameter

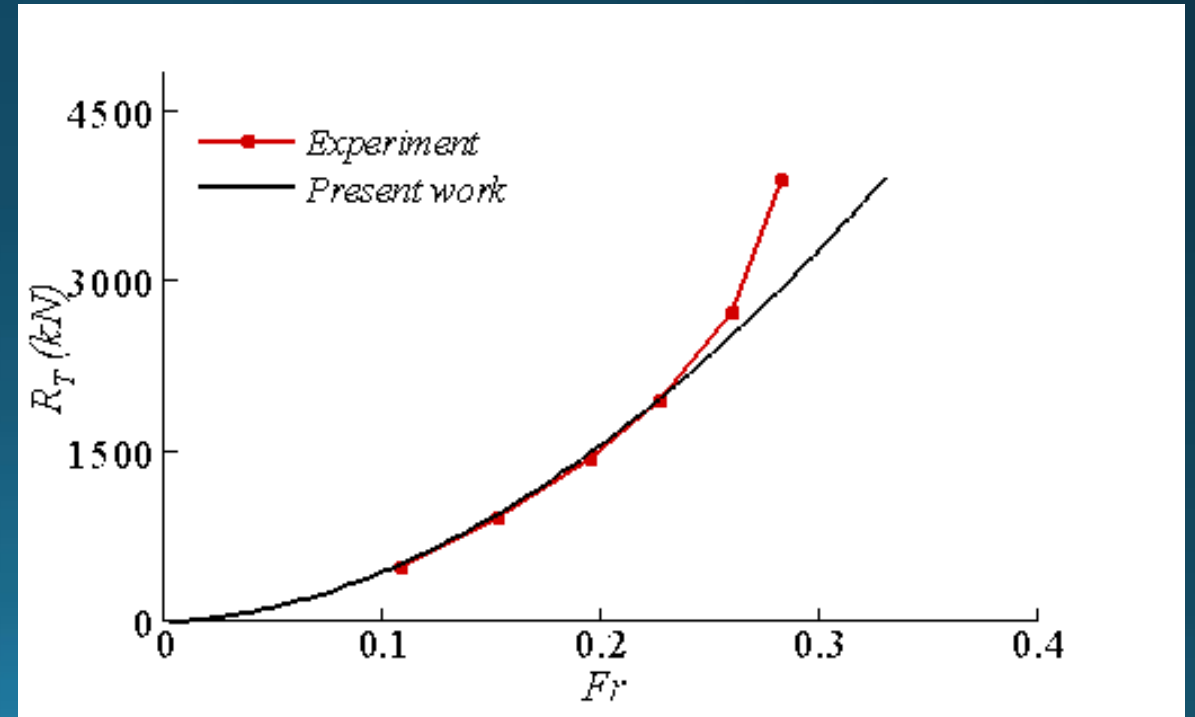
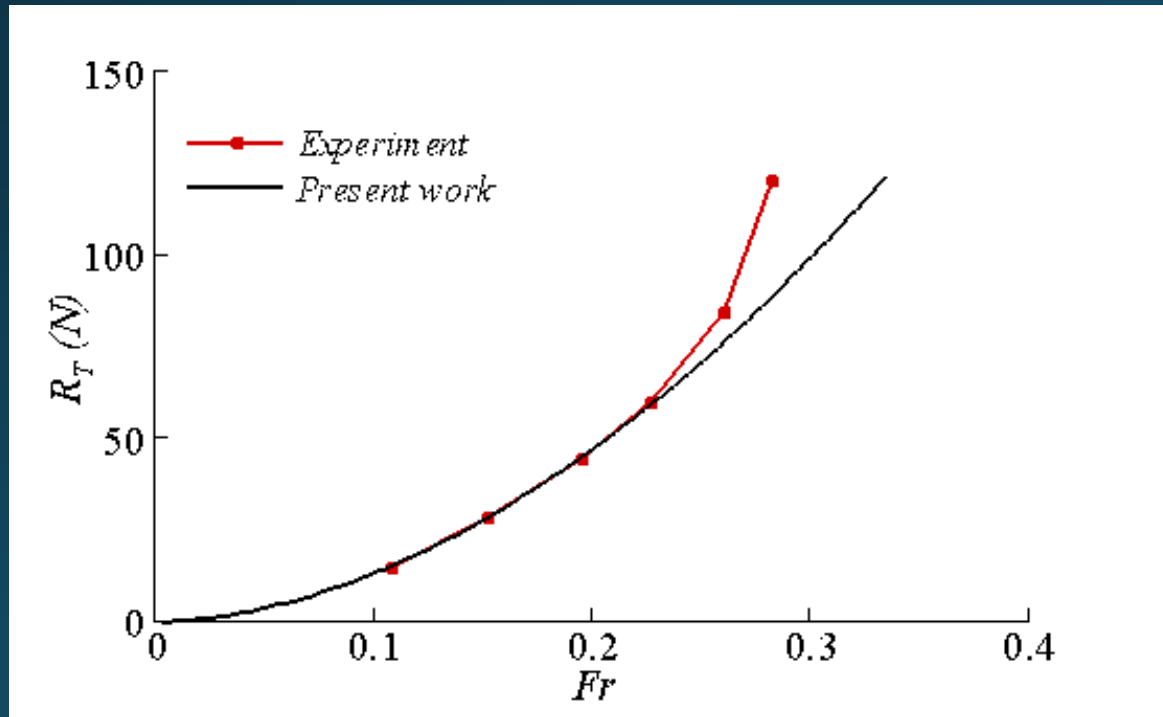
$$\frac{R}{\frac{1}{2} \rho \mathcal{A}_p U^2} = \left(1 + \frac{B}{\mathcal{A}_p} \frac{\sin^2 \beta}{2g} U^2 \right) \sin^2 \beta + 0.072 \frac{(\mathcal{S}_w / \mathcal{A}_p)}{(Re/\sqrt{\nabla})^{1/5}}$$

Applications to Ship Resistance

$$\frac{R}{\frac{1}{2}\rho\mathcal{A}_pU^2} = \left(1 + \frac{B}{\mathcal{A}_p} \frac{\sin^2\beta}{2g} U^2\right) \sin^2\beta + 0.072 \frac{(\mathcal{S}_w/\mathcal{A}_p)}{(Re/\sqrt{\nabla})^{1/5}}$$

$$\sin^2\beta = \frac{1}{1 + (2\xi L/B)^2}$$

MOERI Container Ship $\xi = 0.45$, $\sin^2\beta = 0.0236$ both for model and ship

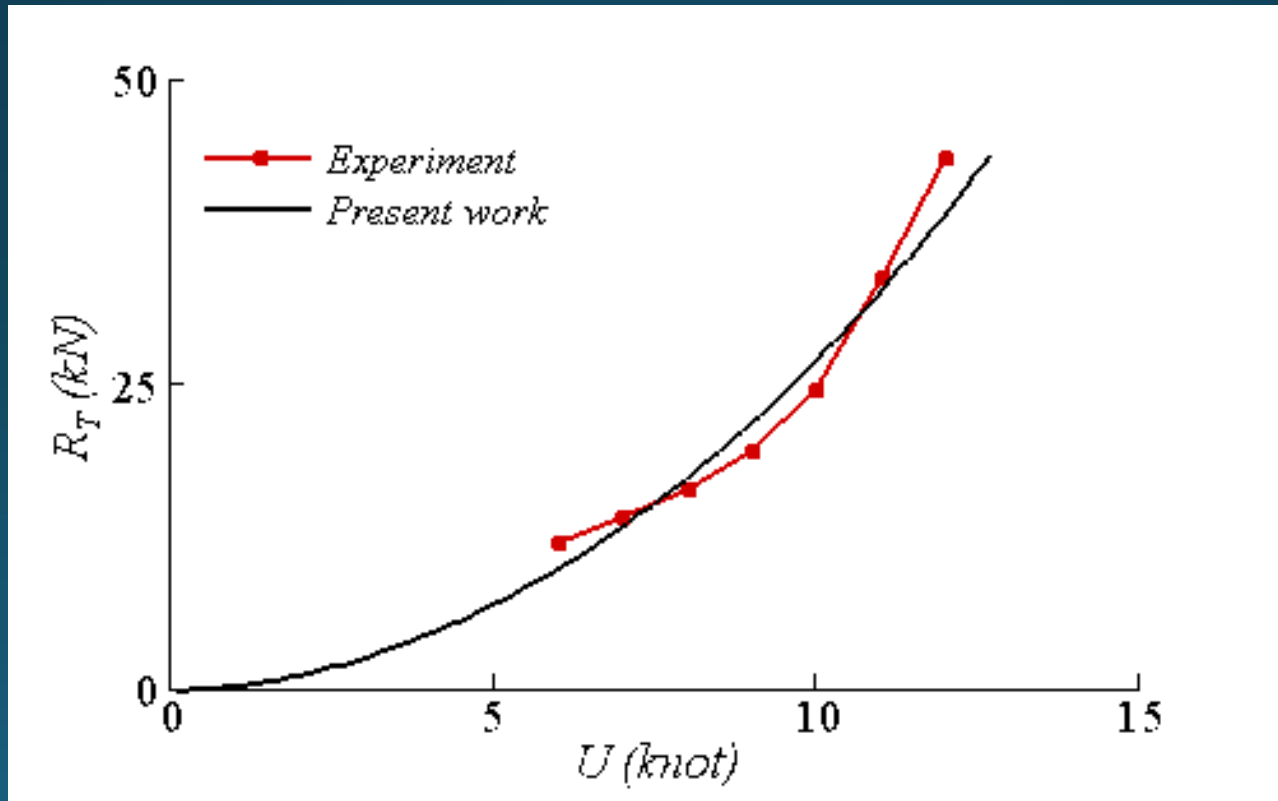


Applications to Ship Resistance

$$\frac{R}{\frac{1}{2}\rho\mathcal{A}_pU^2} = \left(1 + \frac{B}{\mathcal{A}_p} \frac{\sin^2\beta}{2g} U^2\right) \sin^2\beta + 0.072 \frac{(\mathcal{S}_w/\mathcal{A}_p)}{(Re/\sqrt{\nabla})^{1/5}}$$

$$\sin^2\beta = \frac{1}{1 + (2\xi L/B)^2}$$

Passenger – Cargo Coaster $\xi = 0.35, \sin^2\beta = 0.0963$

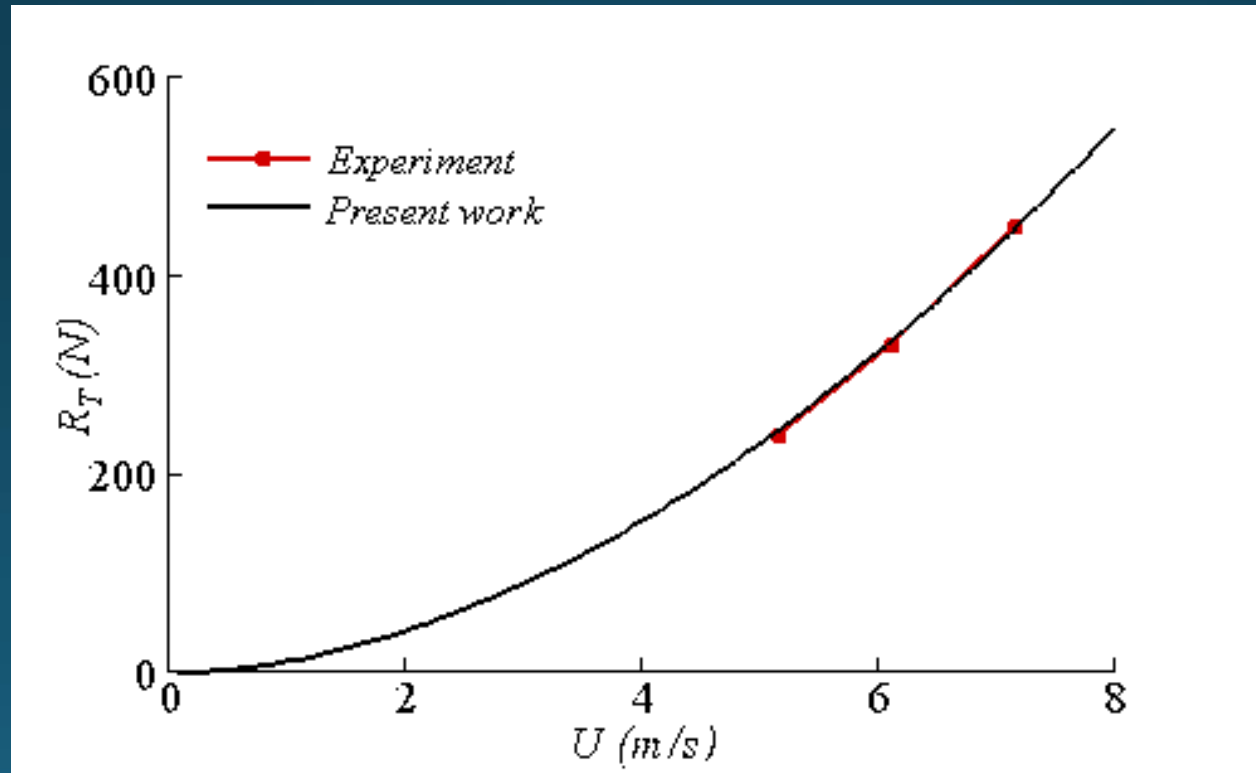


Applications to Ship Resistance

$$\frac{R}{\frac{1}{2}\rho\mathcal{A}_pU^2} = \sin^2\beta + 0.072 \frac{(\mathcal{S}_w/\mathcal{A}_p)}{(Re/\sqrt{\nabla})^{1/5}}$$

$$\sin^2\beta = \frac{1}{1 + (2\xi L/B)^2}$$

DARPA – SUBOFF Submarine $\xi = 0.35, \sin^2\beta = 0.027$



Forces due to current at an angle to a ship

$$\frac{F_x}{\frac{1}{2}\rho\mathcal{A}_{px}u_r|u_r|} = \left(1 + \frac{B}{\mathcal{A}_{px}} \frac{\sin^2\beta_x}{2g} u_r^2\right) \sin^2\beta_x + 0.072 \frac{(\mathcal{S}_w/\mathcal{A}_{px})}{(Re_x/\sqrt{\nabla})^{1/5}}$$
$$\frac{F_y}{\frac{1}{2}\rho\mathcal{A}_{py}v_r|v_r|} = \left(1 + \frac{L}{\mathcal{A}_{py}} \frac{\sin^2\beta_y}{2g} v_r^2\right) \sin^2\beta_y + 0.072 \frac{(\mathcal{S}_w/\mathcal{A}_{py})}{(Re_y/\sqrt{\nabla})^{1/5}}$$

Relative current speeds: $u_r = u - u_c$, $v_r = v - v_c$

$$u_c = -V_c \cos \gamma_c, \quad v_c = V_c \sin \gamma_c$$

Current speed: V_c , Current angle of attack: $\gamma_c = \psi - \beta_c - \pi$

Ship heading angle: ψ , Current direction angle (due North): β_c

$$\mathcal{A}_{px} = \nabla/L, \quad \mathcal{A}_{py} = \nabla/B, \quad Re_x = |u_r|L/\nu, \quad Re_y = |v_r|B/\nu$$

$$\sin^2\beta_x = 1/[1 + (2\xi_x L/B)^2], \quad \sin^2\beta_y = 1/[1 + (2\xi_y B/L)^2]$$

Concluding Remarks

- Wave and current forces acting on a body in water are theoretically formulated from 1-D momentum equation.
- Morison's equation is recovered as a special case from the general formulation.
- A simple ship resistance formula containing a single tuning parameter is derived and the scaling problem inherent to ship resistance calculations is resolved.
- Performance of ship resistance formula is tested against several experimental measurements with satisfactory results.
- Resistance formula for ships is adapted to express forces due to a current at an angle to a ship.